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How to Uncross Some Modular Metrics

Alexander V. Karzanov

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Dedicated to the memory of Walter Deuber

Abstract

Let μ be a metric on a set T, and let c be a nonnegative function on the unordered pairs of elements of a superset $V \supseteq T$. We consider the problem of minimizing the inner product $c \cdot m$ over all semimetrics m on V such that m coincides with μ within T and each element of V is at zero distance from T (a variant of the *multifacility location problem*). In particular, this generalizes the well-known multiterminal (or multiway) cut problem.

Two cases of metrics μ have been known for which the problem can be solved in polynomial time: (a) μ is a modular metric whose underlying graph $H(\mu)$ is hereditary modular and orientable (in a certain sense); and (b) μ is a median metric. In the latter case an optimal solution can be found by use of a cut uncrossing method.

In this paper we generalize the idea of cut uncrossing to show the polynomial solvability for a wider class of metrics μ, which includes the median metrics as a special case. The metric uncrossing method that we develop relies on the existence of retractions of certain modular graphs. On the negative side, we prove that for μ fixed, the problem is NP-hard if μ is non-modular or $H(\mu)$ is non-orientable.

Keywords: location problem, multiterminal (multiway) cut, modular graph

AMS Subject Classification: 05C12, 90C27, 90B10, 57M20

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†On leave from the Institute for System Analysis of the Russian Acad. Sci., Prospect 60 Let Oktyabrya, 117312 Moscow, Russia, email: *sasha@cs.isa.ac.ru*

1 Introduction

We deal with a variant of the *multifacility location problem.* In its setting, there are a finite metric space (T, μ), a finite set X, and a nonnegative function c on the pairs of elements of $T \cup X$. (The elements of T are thought of as the points where the existing facilities are located, the elements of X as new facilities, and $c(x, y)$ as a measure of communication between x to y.) The objective is to place each new facility at a point of T minimizing the sum of values $c(x, y)\mu(x', y')$, where x, y range over the pairs of facilities and x', y' are the points of T where x, y are placed. For a survey on location problems, see, e.g., [14].

This problem can be reformulated in terms of metric extensions. We start with some terminology and notation. A *semimetric* on a set S is a function $d : S \times S \to \mathbf{R}_+$ that establishes *distances* on the pairs of elements (*points*) of S satisfying $d(x, x) = 0$, $d(x, y) = d(y, x)$ and $d(x, y) + d(y, z) \geq d(x, z)$, for all $x, y, z \in S$. We use notation xy for an unordered pair $\{x, y\}$ and usually write $d(xy)$ instead of $d(x, y)$. The set of pairs xy with $x \neq y$ is denoted by E_S. When $d(xy) > 0$ for all $xy \in E_S$, d is a *metric.* We do not distinguish between the (semi)metric d and the (semi)metric space (S, d) and usually deal with only finite (semi)metric spaces. A special case is the *path metric* d^G of a connected graph G, where $d^G(xy)$ is the minimum number of edges of a path in G connecting nodes x and y.

A semimetric m on a set $V \supseteq S$ is said to be an *extension* of d if the restriction (*submetric*) of m to S is just d. Such an m is called a *0-extension* if the distance $m(x, S)$ from each point $x \in V$ to S is zero, i.e., $m(xs) = 0$ for some $s \in S$. Clearly each 0-extension m is uniquely determined by the *0-distance sets* $X_s = \{x \in V : m(xs) = 0\}$, $s \in S$, and these sets give a partition of V when d is a metric.

The above problem is equivalent to the *minimum 0-extension problem* : Given a metric μ on a set T, a superset $V \supseteq T$, and a function $c : E_V \to \mathbf{Z}_+$,

(1.1) *Find a 0-extension m of μ to V with $c \cdot m := \sum(c(e)m(e) : e \in E_V)$ minimum.*

In this paper we extend earlier results on the complexity of (1.1) for fixed metrics μ.

Two classes of metrics μ have been found for which (1.1) is solvable in polynomial time. One class consists of the metrics for which (1.1) becomes as easy as its linear programming relaxation. More precisely, let $\tau = \tau(V, c, \mu)$ denote the minimum $c \cdot m$ in (1.1), and let $\tau^* = \tau^*(V, c, \mu)$ denote the minimum $c \cdot m$ in the problem:

(1.2) *Find an extension m of μ to V with $c \cdot m$ minimum.*

Then $\tau \geq \tau^*$. A metric μ is called *minimizable* if $\tau(V, c, \mu) = \tau^*(V, c, \mu)$ holds for any V and c. Since (1.2) is a linear program whose constraint matrix size is polynomial in $|V|$, (1.2) is solvable in strongly polynomial time. This easily implies that for every minimizable metric μ, on optimal 0-extension in (1.1) can be found in strongly polynomial time as well. The following theorem characterizes the set of minimizable path metrics.

Theorem 1.1 [10] *For a graph H, the metric d^H is minimizable if and only if H is hereditary modular and orientable.*

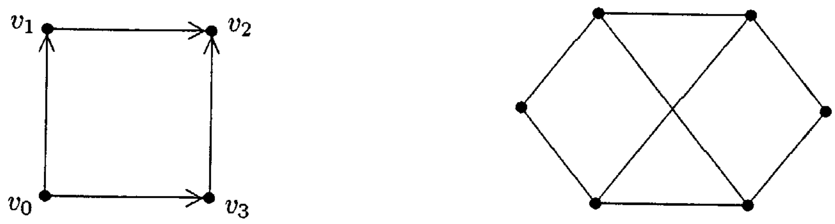

Figure 1: (a) An orientation of a 4-circuit (b) $K_{3,3}^-$

Recall that a metric μ on T is *modular* if every three points $s_0, s_1, s_2 \in T$ have a *median*, a node $z \in T$ satisfying $\mu(s_iz)+\mu(zs_j) = \mu(s_is_j)$ for all $0 \le i < j \le 2$. A graph H is called *modular* if d^H is modular, and *hereditary modular* if every isometric subgraph of H is modular, where a subgraph (or circuit) $H' = (T', U')$ of H is *isometric* if $d^{H'}(st) = d^H(st)$ for all $s, t \in T'$. Every modular graph is bipartite. A graph is called *orientable* if its edges can be oriented so that for any 4-circuit $C = (v_0, e_1, v_1, ..., e_4, v_4 = v_0)$ and $i = 1, 2$, the edge e_i is oriented from v_{i-1} to v_i if and only if the opposite edge e_{i+2} is oriented from v_{i+2} to v_{i+1}; see Fig. 1(a). For example, every bipartite graph with at most five nodes is hereditary modular and orientable. The simplest hereditary modular but not orientable graph is the graph $K_{3,3}^-$ obtained from $K_{3,3}$ by deleting one edge; see Fig. 1(b). Using terminology in [10], we refer to an orientable hereditary modular graph as a *frame*.

Theorem 1.1 is extended to general metrics using the notion of *underlying graph* of μ. This is the least graph $H(\mu) = (T, U(\mu))$ which enables us to restore μ if we know the distances of its edges. Formally, nodes $x, y \in T$ are adjacent in $H(\mu)$ if and only if no other node $z \in T$ lies *between* x and y, i.e., satisfies $\mu(xz) + \mu(zy) = \mu(xy)$. This graph is modular if μ is modular [1].

Theorem 1.2 [3] *A metric μ is minimizable if and only if μ is modular and $H(\mu)$ is a frame.*

Another tractable case involves *median metrics*, the metrics μ with precisely one median for each triplet of points. Chepoi [5] showed that (1.1) with any median metric μ is solvable in strongly polynomial time. A simple alternative method, based on cut uncrossing techniques, is suggested in [10]. Note that a minimizable metric need not be a median one, and vice versa. For example, $d^{K_{2,3}}$ is minimizable but not median, while the path metric of the (skeleton of the) cube is median but not minimizable (the cube is not hereditary modular as it contains an isometric 6-circuit).

In this paper we show the polynomial solvability for a class of modular metrics which includes the median ones as a special case. It uses the notion of orbit graphs that we now introduce. Given a modular graph $H = (T, U)$, two edges are called *mates* if they are opposite in some 4-circuit; when dealing with graphs with possible parallel edges, we refer to such edges as mates as well. Edges e, e' of H are called *projective* if there is a sequence $e = e_0, e_1, \ldots, e_k = e'$ of edges such that

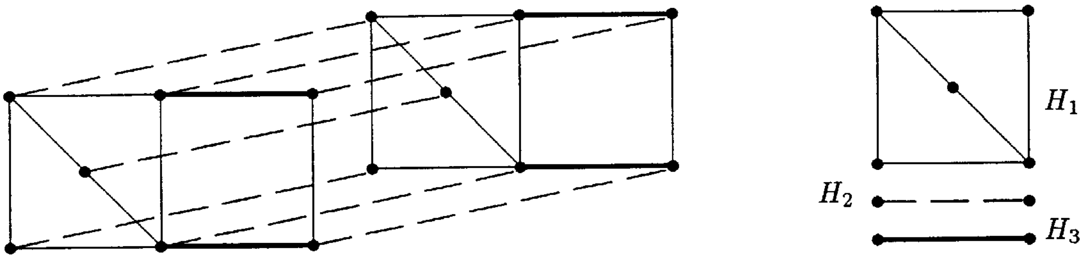

Figure 2: graph H orbit graphs

any two consecutive e_i, e_{i+1} are mates; such a sequence is called *projective* too. A maximal set Q of mutually projective edges is called an *orbit*. Define the *orbit graph* H_Q to be $H//(U-Q)$, where for a graph H' and a subset Z of its edges, $H'//Z$ denotes the graph obtained by contracting Z (i.e., forming H'/Z) and then identifying parallel edges appeared.

The main result in this paper is the following.

Theorem 1.3 *Let μ be a modular metric with underlying graph $H = (T, U)$, and let for each orbit Q of H,*

(i) *the orbit graph H_Q is a frame, and*

(ii) *H_Q is isomorphic to some subgraph of the graph (T, Q).*

Then (1.1) can be solved in strongly polynomial time.

We shall explain later that each orbit graph of a frame is a frame, and each orbit graph of a median graph is K_2, which is a trivial case of frames. Since condition (ii) in Theorem 1.3 obviously holds when H_Q is K_2, this theorem generalizes the above result for median metrics. On the other hand, the set of metrics μ in this theorem does not contain some minimizable metrics since there are frames H for which (ii) is not valid. One can show that (ii) holds when each orbit graph is either K_2 or $K_{2,r}$ for $r \geq 3$, the simplest cases of frames with one orbit. Figure 2 illustrates the graph H with three orbits, drawn by thin, dashed and bold lines, whose orbit graphs are $H_1 \simeq K_{2,3}$, $H_2 \simeq K_2$ and $H_3 \simeq K_2$.

The proof of Theorem 1.3 will involve a number of reductions. One of them is to show that this theorem can be derived from Theorem 1.1 and Theorem 1.4 below that claims the existence of a retraction for certain graphs. Here a *retraction* of a bipartite graph $K = (V(K), E(K))$ onto its subgraph $K' = (V(K'), E(K'))$ is meant to be a mapping $\gamma : V(K) \to V(K')$ which is identical on $V(K')$ (i.e., $\gamma(v) = v$ for all $v \in V(K')$) and brings each edge of K to an edge of K' (i.e., $\gamma(u)\gamma(v) \in E(K')$ for all $uv \in E(K)$). Suppose K is the Cartesian product $H_1 \times \ldots \times H_k$ of graphs $H_i = (T_i, U_i)$, $i = 1, \ldots, k$, i.e., $V(K) = T_1 \times \ldots \times T_k$ and nodes $(s_1, \ldots, s_k)$ and $(t_1, \ldots, t_k)$ of K are adjacent if and only if there is $i \in \{1, \ldots, k\}$ such that $s_i t_i \in U_i$ and $s_j = t_j$ for $j \neq i$. For a subgraph K' of K and $i \in \{1, \ldots, k\}$, an *i-layer* of K' is a maximal subgraph of K' induced by nodes $(t_1, \ldots, t_k)$ with $t_1, \ldots, t_{i-1}, t_{i+1}, \ldots, t_k$ fixed.

Theorem 1.4 *Let K be the Cartesian product of frames $H_i = (T_i, U_i)$, $i = 1, \dots, k$. Let K' be an isometric subgraph of K such that K' is modular and for each $i = 1, \dots, k$, some of the i-layers of K' is isomorphic to H_i. Then there exists a retraction of K onto K'.*

(Note that K' is not an absolute retract in general, i.e., K' need not admit retraction of *any* bipartite graph which contains K' as an isometric subgraph. Necessary and sufficient conditions on a bipartite graph to be an absolute retract are given in [4].) In our case, the role of graphs H_i and K' in Theorem 1.4 will play the graphs H_Q and H in Theorem 1.3, using the important observation that H has a canonical isometric embedding in the Cartesian product K of its orbit graphs. It turns out that Theorem 1.4 can be rather easily reduced to its special case with $k = 2$; moreover, such a reduction takes place for arbitrary modular graphs $H_1, \dots, H_k$. To show the existence of a retraction for this special case, with H_1, H_2 frames, is the core of the whole proof of Theorem 1.3. Such a retraction is just behind our "metric uncrossing operation", an analogue of the cut uncrossing operation for 0-extensions of the corresponding orbit metrics (when both H_1, H_2 are K_2, the retraction is evident and it induces the uncrossing of two cuts, as we explain later).

Next we deal with intractable cases. When $\mu = d^{K_p}$, (1.1) turns into the *minimum multiterminal* (or *multiway*) *cut problem*, which is strongly NP-hard already for $p = 3$ [6]. That result has been generalized to a larger set of path metrics.

Theorem 1.5 [10] *For a fixed graph H, problem (1.1) with $\mu = d^H$ is strongly NP-hard if H is non-modular or non-orientable.*

We extend this theorem as follows.

Theorem 1.6 *For a fixed rational-valued metric μ, (1.1) is strongly NP-hard if μ is non-modular or if the underlying graph $H(\mu)$ is non-orientable.*

The structure of this paper is as follows. Section 2 demonstrates some basic properties of modular metrics and graphs and their orbit graphs. Section 3 describes our approach to proving Theorem 1.3 and is aimed to explain why this theorem reduces to Theorem 1.4 with $k = 2$. The desired retraction is constructed in Section 4, using combinatorial arguments and relying on some result concerning the tight spans of minimizable path metrics from [10]. The construction also relies on a key lemma proved in Section 5. The proof of Theorem 1.6 is given in Section 6.

By technical reasons, in problems (1.1) and (1.2) we will sometimes admit $\mu(st) = 0$ for distinct s, t and may speak about minimizable semimetrics rather than metrics; this does not affect the problem area in essense. The sets of extensions and 0-extensions of a (semi)metric μ to a set V are denoted by $\mathrm{Ext}(\mu, V)$ and $\mathrm{Ext}^0(\mu, V)$, respectively.

2 Modular metrics, modular graphs, and orbits

By a *u–v path on* a set V we mean any sequence $P = (x_0, x_1, \dots, x_k)$ of elements of V with $x_0 = u$ and $x_k = v$. For a semimetric m on V, the *m-length* $m(P)$ of P is $m(x_0x_1) + \dots + m(x_{k-1}x_k)$, and P is called *shortest* w.r.t. m, or *m-shortest*, if $m(P) = m(uv)$. If each pair $e_i = x_{i-1}x_i$ is an edge of a graph $G = (V, E)$, then $P = (x_0, e_1, x_1, \ldots, e_k, x_k)$ is a *path in* G, and we say that P is *G-shortest* if its length $|P| := k$ is equal to $d^G(uv)$. When it is not confusing, we abbreviate $P = x_0x_1 \ldots x_k$. Given nonnegative *lengths* $\ell(e)$ of the edges $e \in E$, we denote by $d^{G,\ell}(xy)$ the minimum length $\ell(P) = \sum(\ell(x_{i-1}x_i) : i = 1, \ldots, k)$ of a path $P = x_0x_1 \ldots x_k$ connecting nodes x and y in G (the *path (semi)metric* for G, ℓ). From the definition of the underlying graph $H(\mu)$ of a metric μ it follows that $\mu = d^{H(\mu),\ell}$ for the restriction ℓ of μ to the edges of $H(\mu)$.

Bandelt [1] showed useful relations between modular graphs and metrics. They can be stated in terms of orbits as follows (cf. [11]).

(2.1) For an orbit Q of a modular graph $H = (T, U)$ and nodes $u, v \in T$, if P is a shortest u–v path and P' is a u–v path in H, then $|P \cap Q| \leq |P' \cap Q|$; in particular, $|P \cap Q| = |P' \cap Q|$ if both P, P' are shortest.

(2.2) For a modular metric μ, the graph $H(\mu)$ is modular and μ is *orbit-invariant*, i.e., it is constant on the edges of each orbit of $H(\mu)$.

(2.3) For a modular graph $H = (T, U)$ and an orbit-invariant function $\ell : U \to \mathbf{R}_+$, the semimetric $\mu = d^{H,\ell}$ is modular, $\mu(e) = \ell(e)$ for all $e \in U$, and every H-shortest path is μ-shortest; moreover, if, in addition, ℓ is positive, then $H = H(\mu)$, and the metrics d^H and μ have the same sets of shortest paths.

Note that μ need not be modular when $H(\mu)$ is modular. (Properties (2.2) and (2.3) are easily derived from (2.1). The latter can be seen as follows (a sketch). Let w be the node of P following u. One may assume P' is simple and all intermediate nodes x of P' are different from w. Since P is shortest and H is bipartite, some node x of P' satisfies $d^H(wx) - 1 = d^H(wy) = d^H(wz)$, where y, z are the neighbours of x in P'. Take a median x' of y, z, w. Then $x'y$ and $x'z$ are edges of H projective to xz and xy, respectively. Therefore, the path P'' obtained from P' by replacing x by x' obeys $|P'' \cap Q| = |P' \cap Q|$, and we can apply induction since the sum of distances from w to the nodes of P'' is less than the corresponding sum for P', in view of $d^H(wx') = d^H(wx) - 2$.)

By (2.3), every modular graph is the underlying graph for the class of modular metrics determined by positive orbit-invariant functions on its edges, and all these metrics have the same sets of shortest paths. This fact will often allow us to work with modular graphs rather than modular metrics.

Consider a modular graph $H = (T, U)$, and let $Q_1, \ldots, Q_k$ be the orbits of H. Let χ^S denote the incidence vector of a subset $S \subseteq U$, i.e. $\chi^S(e) = 1$ for $e \in S$, and 0 for $e \in U - S$. Any modular metric μ with $H(\mu) = H$ is representable as

$$\mu = h_1\mu_1 + \ldots + h_k\mu_k, \tag{2.4}$$

where $\mu_i = d^{H,\ell_i}$ for $\ell_i = \chi^{Q_i}$ and $h_i = \mu(e)$ for $e \in Q_i$ (h_i is well-defined by (2.2)). Indeed, for any $s, t \in T$, a shortest s–t path P in H is shortest for each of $\mu, \mu_1, \ldots, \mu_k$, and μ_i coincides with ℓ_i on U, by (2.3). Therefore,

$$\mu(st) = \mu(P) = h_1\ell_1(P) + \ldots + h_k\ell_k(P) = h_1\mu_1(st) + \ldots + h_k\mu_k(st),$$

as required. When all h_i's are ones, (2.4) is specified as

$$d^H = \mu_1 + \ldots + \mu_k. \tag{2.5}$$

Some properties of H preserve under contraction of orbits. Let $H' = (T', U')$ be the graph H/Q_1. We identify the edges in $U - Q_1$ with their images in H' and denote by $\varphi(x)$ (resp. $\varphi(P)$) the image in H' of a node x (resp. a path P) of H. By (2.3) applied to the orbit-invariant function $\ell = \chi^{U-Q_1}$,

(2.6) if P is a shortest path of H, then $\varphi(P)$ is a shortest path of H'.

Therefore, if v is a median of nodes x, y, z in H, then $\varphi(v)$ is a median of $\varphi(x), \varphi(y), \varphi(z)$ in H'. This implies that H' is modular.

Statement 2.1 *$Q_2, \ldots, Q_k$ are the orbits of H'.*

Proof. Obviously, mates $e, e' \in U - Q_1$ of H remain mates in H', i.e., they are either opposite in a 4-circuit or parallel. This implies that each set Q_i $(i > 1)$ is entirely included in some orbit of H'. To see the reverse inclusion, consider a 4-circuit $C = (v_0, e_1, v_1, \ldots, e_4, v_4 = v_0)$ of H', and let L_j denote the path $(v_j, e_{j+1}, v_{j+1}, e_{j+2}, v_{j+2})$ for $j = 0, \ldots, 3$ (taking indices modulo 4). Each L_j is a shortest path since H' is bipartite (as being modular). Choose $x_0 \in \varphi^{-1}(v_0)$ and $x_2 \in \varphi^{-1}(v_2)$, and let P_0 and P_2 be two x_0–x_2 paths of H whose images in H' are L_0 and the reverse to L_2, respectively. Let P be a shortest x_0–x_2 path in H. Then $|\varphi(P)| = |L_0| = |L_2| = 2$. This together with (2.1) (applied to P and $P' = P_0, P_2$) implies $|P \cap Q_i| = |L_0 \cap Q_i| = |L_2 \cap Q_i|$ for $i = 2, \ldots, k$. Similarly, $|L_1 \cap Q_i| = |L_3 \cap Q_i|$ for each i. These equalities are possible only if each pair of mates in C belongs to the same set Q_i. Similar arguments are applied to parallel edges of H' (if any). ■

Repeatedly applying this statement to orbits of H, we obtain the following.

Corollary 2.2 *For any $I \subseteq \{1, \ldots, k\}$, the graph $H/(\cup_{i \in I} Q_i)$ is modular and its orbits are the sets Q_j for $j \in \{1, \ldots, k\} - I$. In particular, each orbit graph H_Q of a modular graph $H = (T, U)$ is modular and has only one orbit, which is obtained by identifying parallel edges in $H/(U - Q)$.*

Next we explain that each orbit graph of $H(\mu)$ is K_2 when μ is a median metric; this follows from properties of median graphs revealed by Mulder and Schrijver [13]. Since μ and $H(\mu)$ have the same sets of shortest paths (by (2.2) and (2.3)), a point v is a median of points x, y, z for μ if and only if v is a median of this triplet for $d^{H(\mu)}$. So $d^{H(\mu)}$ is a median metric, which means that $H(\mu)$ is a *median graph*. It is shown in [13] that

(2.7) $H = (T, U)$ is a median graph if and only if $d^H = \mu_1 + \ldots + \mu_k$, where each μ_i is the cut metric corresponding to a bi-partition $\{A_i, T - A_i\}$ of T (i.e., $\mu_i(st) = 1$ if $|\{s, t\} \cap A_i| = 1$, and 0 otherwise), and the family $\mathcal{F} = \{A_1, \ldots, A_k, T - A_1, \ldots, T - A_k\}$ satisfies the *Helly property* (i.e., any subfamily $\mathcal{F}' \subseteq \mathcal{F}$ has a nonempty intersection provided that each two members of $\mathcal{F}'$ meet).

Let Q_i be the set of edges of H connecting A_i and $T - A_i$; clearly $Q_1, \ldots, Q_k$ are pairwise disjoint. These sets are precisely the orbits of H. Indeed, in view of $d^H = \mu_1 + \ldots + \mu_k$, a shortest path of H is μ_i-shortest for each i. This easily implies that: (i) the subgraphs of H induced by A_i and by $T - A_i$ are connected, and (ii) Q_i is a matching. ([13] shows the sharper property that H is median if and only if H is modular and has a cutset edge colouring.) Since Q_i is simultaneously a cut and a matching, if e, e' are mates in H and $e \in Q_i$, then $e' \in Q_i$. So each orbit Q of H is included in some Q_i. Suppose $Q \neq Q_i$. Then the subgraph $(T, U - Q)$ is connected, by (i) above, whence the semimetric $\mu' = d^{H,\ell}$ for $\ell = \chi^Q$ is identically zero. This is impossible because μ' coincides with ℓ on U, by (2.3). Thus, Q_i is an orbit. Now (i) implies that $H/(U - Q_i)$ is a tuple of parallel edges, and we conclude that each orbit graph of H is K_2.

As mentioned in the Introduction, our approach to solving problem (1.1) with a metric figured in Theorem 1.3 generalizes the cut uncrossing method for median metrics μ. We now briefly describe that method, referring the reader for details to [10, Sec. 5].

Given a median metric μ on T, a set $V \supseteq T$ and a function $c : E_V \to \mathbf{Z}_+$, represent μ as in (2.4), where each μ_i is the cut metric corresponding to a bi-partition $\{A_i, T - A_i\}$ of T as in (2.7). For $i = 1, \ldots, k$, find a bi-partition $\{X_i, \overline{X}_i\}$ of V such that $X_i \cap T = A_i$ and $\sum(c(xy) : x \in X_i \not\ni y)$ is minimum (a *minimum cut* separating A_i and $T - A_i$). Let $\mathcal{X} = \{X_1, \ldots, X_k, \overline{X}_1, \ldots, \overline{X}_k\}$, and let $m = h_1 m_1 + \ldots + h_k m_k$, where m_i is the cut metric on V corresponding to $\{X_i, \overline{X}_i\}$. Choose a pair $Y, Z \in \mathcal{X}$ such that $Y \cap Z \cap T = \emptyset$ but Y, Z meet, and make "uncrossing" by replacing Y, Z in $\mathcal{X}$ by $Y' = Y - Z$ and $Z' = Z - Y$ (taking into account that $\{Y', \overline{Y'}\}$ induces a minimum cut separating $Y \cap T$ and $\overline{Y} \cap T$, and $\{Z', \overline{Z'}\}$ induces a minimum cut separating $Z \cap T$ and $\overline{Z} \cap T$). Iterate until the current family $\mathcal{X}'$ has no such pair Y, Z, i.e., $Y \cap Z \cap T = \emptyset$ implies $Y \cap Z$. Using the Helly property for $\mathcal{F}$ in (2.7), one can see that the corresponding metric $m' = h_1 m'_1 + \ldots + h_k m'_k$ is a 0-extension. Moreover, the fact that each m'_i is induced by a minimum cut implies that m' is optimal. One shows that the number of iterations does not exceed $|T|^2|V|$ (in fact, one can arrange a process consisting of only $O(k^2)$ uncrossing operations).

It turns out that the Helly property for median graphs exhibited in (2.7) is extended to general modular graphs. More precisely, for a modular graph $H = (T, U)$ with orbits $Q_1, \ldots, Q_k$, let $H_i = (T_i, U_i)$ stand for H_{Q_i}, and define $\pi_i = \{A_i(t) : t \in T_i\}$ to be the partition of T where each member $A_i(t)$ is the node set of the component of $(T, U - Q_i)$ whose contraction creates the node t of H_i. Each $A_i(t)$ is just the corresponding maximal 0-distance set of the metric $\mu_i = d^{H,\ell_i}$ as in (2.5). We assert that

(2.8) the family $\pi = \pi(H)$ of subsets of T occurring in $\pi_1, \ldots, \pi_k$ has the Helly property.

Indeed, each set $A \in \pi$ is *convex*, i.e., for any $x, y \in A$, each node on a shortest x–y path P of H belongs to A. To see this, assume $A \in \pi_i$. Then $\mu_i(xy) = 0$, and therefore, $\ell_i(P) = 0$ (by (2.3)). So all nodes of P are in A, as required. Now the result follows from the simple fact that the family $\overline{\pi}$ of convex node sets of an arbitrary modular graph has the Helly property. (This is shown by induction on n, considering a collection $\pi' = \{A^1, \ldots, A^n\}$ of $n \geq 3$ members of $\overline{\pi}$ such that any $n-1$ of them meet. For $i = 1, 2, 3$, choose an element x_i contained in all sets in π' except possibly

A^i. Let z be a median of x_1, x_2, x_3. For each $A^j \in \pi'$, at least two of x_1, x_2, x_3 belong to A^j, hence $z \in A^j$ by the convexity. Thus, the members of π' have a common element.)

In conclusion of this section we show the hereditary property for orbit graphs of frames.

Statement 2.3 *Let $H = (T, U)$ be a frame, and let Z be the union of some orbits of H. Then H/Z is a frame. In particular, each orbit graph of H is a frame.*

Proof. One can try to prove directly that the graph $H/Z =: H' = (T', U')$ is hereditary modular and orientable. We, however, can use Theorem 1.2 and standard compactness arguments to show that $d^{H'}$ is minimizable. Then H' is a frame by Theorem 1.1.

More precisely, define the semimetric μ on T to be $d^{H,\ell}$ for $\ell = \chi^{U-Z}$. Consider $V' \supseteq T'$ and $c' : E_{V'} \to \mathbf{Z}_+$. We have to show that $\tau(V', c', d^{H'}) = \tau^*(V', c', d^{H'})$. Let $V = V' \cup T$ (assuming $V' \cap T = T'$) and define $c(e) = c'(e)$ for $e \in E_{V'}$, and $c(e) = 0$ for $e \in E_V - E_{V'}$. Clearly $\tau(V, c, \mu) = \tau(V', c', d^{H'})$ and $\tau^*(V, c, \mu) = \tau^*(V', c', d^{H'})$. So it is suffices to prove $\tau(V, c, \mu) = \tau^*(V, c, \mu)$.

To see the latter, consider the infinite sequence $d_1, d_2, \ldots$ of approximations for μ, where d_i is d^{H,ρ_i} with $\rho_i(e) = 1$ for $e \in U - Z$, and $\rho_i(e) = 1/i$ for $e \in Z$. Since H is modular and ρ_i is positive and orbit-invariant, $H = H(d_i)$ for each i by (2.3). So d_i is minimizable (by Theorem 1.2), whence $\tau(V, c, d_i) = \tau^*(V, c, d_i)$. When i grows, $\tau(V, c, d_i)$ tends to $\tau(V, c, \mu)$ (since the number of partitions of V is finite). Also $\tau^*(V, c, d_i)$ tends to $\tau^*(V, c, \mu)$, because of the obvious fact that for any $m \in \mathrm{Ext}(\mu, V)$, there exists $m' \in \mathrm{Ext}(d_i, V)$ such that $|m'(e) - m(e)| \leq |V|/i$ for each $e \in E_V$. Thus, $\tau(V, c, \mu) = \tau^*(V, c, \mu)$, as required. ∎

3 Reduction to the case of two orbits, and uncrossing method

In this section we describe our approach to proving Theorem 1.3. A majority of arguments below are applicable to general modular metrics, and unless explicitly said otherwise, we assume that μ is an arbitrary modular metric on a set T.

Let $H = (T, U)$ be the underlying graph $H(\mu)$ of μ with orbits $Q_1, \ldots, Q_k$. As before, for $i = 1, \ldots, k$, $H_i = (T_i, U_i)$ stands for H_{Q_i}, ℓ_i for χ^{Q_i}, μ_i for d^{H,ℓ_i}, and $\pi_i = \{A_i(t) : t \in T_i\}$ for the corresponding partition of T defined in the previous section. We formally identify each $t \in T_i$ with some element of $A_i(t)$, which allows us to speak of μ_i as a 0-extension of d^{H_i} to T.

For the given μ, consider an instance of the minimum 0-extension problem with $V \supseteq T$ and $c : E_V \to \mathbf{Z}_+$. By (2.4), any 0-extension m of μ to V is representable as

$$m = h_1 m_1 + \ldots + h_k m_k, \tag{3.1}$$

where each m_i is the 0-extension of μ_i to V, defined by

(3.2) $\quad m_i(xy) = \mu_i(st)$ for $x, y \in V$ and $s, t \in T$ with $m(xs) = m(yt) = 0$.

Then $c \cdot m = c \cdot (h_1 m_1) + \ldots + c \cdot (h_k m_k)$ and $c \cdot m_i \geq \tau(V, c, \mu_i)$ for each i. Taking as m an optimal 0-extension for V, c, μ, we conclude that

$$\tau(V, c, \mu) \geq h_1 \tau(V, c, \mu_1) + \ldots + h_k \tau(V, c, \mu_k). \tag{3.3}$$

In particular, this is valid for $h_1 = \ldots = h_k = 1$ and $\mu = d^H$. We say that H is *orbit-additive* if

$$\tau(V, c, d^H) = \tau(V, c, \mu_1) + \ldots + \tau(V, c, \mu_k) \tag{3.4}$$

holds for any V and c. Such an H has a sharper property.

Statement 3.1 *Let H be orbit-additive. Then for any numbers $h_1, \ldots, h_k \geq 0$, the semimetric $\mu = d^{H,\ell}$ with $\ell = h_1\ell_1 + \ldots + h_k\ell_k$ satisfies*

$$\tau(V, c, \mu) = h_1\tau(V, c, \mu_1) + \ldots + h_k\tau(V, c, \mu_k). \tag{3.5}$$

Moreover, if m is an optimal 0-extension for V, c, d^H and $m_1, \ldots, m_k$ are defined as in (3.2), then $m' = h_1m_1 + \ldots + h_km_k$ is an optimal 0-extension for V, c, μ.

Proof. Since $\tau(V, c, d^H) = c \cdot m = c \cdot m_1 + \ldots + c \cdot m_k$, (3.4) implies $c \cdot m_i = \tau(V, c, \mu_i)$ for each i. Clearly $m' \in \mathrm{Ext}^0(\mu, V)$. Therefore, $\tau(V, c, \mu) \leq c \cdot m' = h_1\tau(V, c, \mu_1) + \ldots + h_k\tau(V, c, \mu_k)$, yielding $\tau(V, c, \mu) = c \cdot m'$ and (3.5), in view of (3.3). ■

Because of (3.5), problem (1.1) for a metric μ whose underlying graph H is orbit-additive becomes as easy as that for the path metrics of orbit graphs of H. Indeed, to compute $\tau(V, c, \mu)$ is reduced to finding the numbers $\tau(V, c, \mu_i)$. Moreover, once there is a subroutine to compute $\tau(V', c', \mu)$ for arbitrary V', c', we can find an optimal 0-extension for the given μ, V, c by applying this subroutine $O(|T||V|)$ times (similarly to the case of minimizable metrics μ, mentioned in the Introduction).

In turn, $\tau(V, c, \mu_i)$ is equal to $\tau(V_i, c_i, d^{H_i})$, where V_i and c_i arise by shrinking the sets $A_i(t)$ in the partition π_i of T to the nodes $t \in T_i$. Formally, $V_i = (V - T) \cup T_i$, $c_i(xy) = c(xy)$ for $x, y \in V - T$, $c_i(xt) = c(\{x\}, A_i(t))$ for $x \in V - T, t \in T_i$, and $c_i(st) = c(A_i(s), A_i(t))$ for $s, t \in T_i$, where $c(A, B)$ denotes $\sum(c(xy) : x \in A, y \in B)$ for $A, B \subseteq V$.

In light of the above discussion, Theorem 1.3 would follow from Theorem 1.1 and the property that if H is as in the hypotheses of Theorem 1.3, then

(3.6) H is orbit-additive.

Remark 1. The property of being orbit-additive is immediate in two cases of modular graphs H. Given V, c, let m_i be an optimal 0-extension for V, c, μ_i, and let $m = m_1 + \ldots + m_k$. By (2.5), $m \in \mathrm{Ext}(d^H, V)$. (i) If H is a frame, then (3.4) holds since $\tau(V, c, d^H) = \tau^*(V, c, d^H) \leq c \cdot m = \tau(V, c, \mu_1) + \ldots + \tau(V, c, \mu_k) \leq \tau(V, c, d^H)$. (ii) If H is isomorphic to the Cartesian product of $H_1, \ldots, H_k$, then m is already a 0-extension of d^H, yielding (3.4); cf. [12].

We further explain that (3.6) would follow from the existence of a retraction onto H of the Cartesian product $K = K(H)$ of the orbit graphs $H_1, \ldots, H_k$ of H (see the Introduction for needed definitions). We will use notation z_i for ith coordinate (component) of a point $z \in V(K)$. Since each H_i is bipartite, so is K. For $v \in T$, define

(3.7) $\phi(v)$ to be $z \in V(K)$ such that $v \in A_i(z_i)$ for $i = 1, \ldots, k$.

Statement 3.2 *For any $u, v \in T$, $d^H(uv) = d^K(\phi(u)\phi(v))$.*

Proof. Let $\phi(u) = s$ and $\phi(v) = t$. We have $d^K(st) = d^{H_1}(s_1t_1) + \ldots + d^{H_k}(s_kt_k)$. Consider a shortest u–v path P in H, and for $i = 1, \ldots, k$, let P_i be the image of P in H_i. Then $|P| = |P_1| + \ldots + |P_k|$, and each P_i is a shortest path, by (2.6). By (3.7), $u \in A_i(s_i)$ and $v \in A_i(t_i)$, so s_i, t_i are the ends of P_i and $|P_i| = d^{H_i}(s_it_i)$. Therefore, $|P| = d^K(st)$. ∎

Thus, ϕ induces an isometric embedding of H into K, called the *canonical* embedding of H. We extend ϕ to the edges of H and, when no confusion can arise, identify H with the subgraph $\phi(H)$ of K. In particular, ϕ is injective; in other words,

(3.8) for $z \in V(K)$, the subset $A_1(z_1) \cap \ldots \cap A_k(z_k)$ of T consists of a single element (namely, $\phi^{-1}(z)$) if $z \in \phi(T)$, and is empty otherwise.

An elementary property of a retraction of a (bipartite) graph $G = (V, E)$ onto its subgraph $G' = (V', E')$ is that γ turns every path of G into a path of G'. This implies that $d^G(xy) - d^{G'}(\gamma(x)\gamma(y))$ is a nonnegative even integer for any $x, y \in V$. Therefore, γ is *non-expansive* (does not increase the distances) and preserves the distance *parity.*

Statement 3.3 *A modular graph H is orbit-additive if there exists a retraction of $K = K(H)$ onto H.*

Proof. Given V, c, for each $i = 1, \ldots, k$, take an optimal 0-extension m_i for V, c, μ_i, and form the extension $m = m_1 + \ldots + m_k$ of d^H to V. Assuming there exists a retraction γ of K onto H, we construct a *0-extension* m' of d^H to V such that $m' \leq m$. This will imply (3.4) since $\tau(V, c, \mu) \leq c \cdot m' \leq c \cdot m$ and $c \cdot m = \tau(V, c, \mu_1) + \ldots + \tau(V, c, \mu_k)$. For $z \in V(K)$, define

$$\begin{aligned} X_i(z_i) &= \{x \in V : m_i(xv) = 0 \text{ some } v \in A_i(z_i)\}, \quad i = 1, \ldots, k; \\ X_z &= X_1(z_1) \cap \ldots \cap X_k(z_k). \end{aligned} \tag{3.9}$$

The mapping $\omega : V \to V(K)$, defined by $\omega(x) = z$ for $x \in X_z$, isometrically embeds (V, m) in $(V(K), d^K)$. Indeed, for $x \in X_z$ and $y \in X_{z'}$, we have

$$m(xy) = m_1(xy) + \ldots + m_k(xy) = d^{H_1}(z_1z_1') + \ldots + d^{H_k}(z_kz_k') = d^K(zz').$$

Also $\omega(v) = v$ for each $v \in T$ (cf. (3.7)), i.e., ω is identical on the node set of the graph H embedded in K by ϕ. The sets X_z give a partition of V, and if it happens that for each *nonempty* set X_z, the set $A_1(z_1) \cap \ldots \cap A_k(z_k)$ is nonempty too (thus consisting of a single node, by (3.8)), then m is already a 0-extension. In general, define the semimetric m' on V by

$$m'(xy) = d^H(\gamma(\omega(x))\gamma(\omega(y))) \quad \text{for } x, y \in V.$$

Then m' is a 0-extension of d^H (corresponding to the partition $\{\omega^{-1}\gamma^{-1}(t) : t \in T\}$). Now the fact that γ is non-expansive while ω is isometric implies $m' \leq m$, as required. ∎

One can see that for each orbit Q_i, the components of the graph (T, Q_i) are just the i-layers of H (canonically embedded in K by ϕ). Thus, condition (ii) in Theorem 1.3 says that each orbit graph H_i is isomorphic to some of the i-layers of H, and now summing up the above reasonings, we conclude that Theorem 1.3 is implied by Theorem 1.4.

So it remains to prove Theorem 1.4. For convenience we denote K' by $H = (T, U)$. Note that now any graph H_i may have more than one orbit, but this is not important for us. First of all we explain that it suffices to consider the case $k = 2$ (in the reduction below we only use the fact that each H_i is modular rather than H_i is a frame).

Let $1 \le i < j \le k$ and $K_{ij} = H_i \times H_j$. Define $H_{ij} = (T_{ij}, U_{ij})$ to be the projection of H to K_{ij}, i.e., $T_{ij} = \{(z_i, z_j) : z \in T\}$ and $U_{ij} = \{(z_i, z_j)(z'_i, z'_j) : zz' \in U,\ z_p = z'_p \text{ for } p \ne i, j\}$. (When H is as in Theorem 1.3, H_{ij} is isomorphic to the "two-orbit graph" $H//(U - Q_i - Q_j)$.) Suppose a retraction γ_{ij} of K_{ij} onto H_{ij} exists for each pair i, j. Define the mapping $\psi_{ij} : V(K) \to V(K)$ by $\psi_{ij}(z) = z'$, where $(z'_i, z'_j) = \gamma_{ij}(z_i, z_j)$ and $z'_p = z_p$ for $p \ne i, j$. Clearly ψ_{ij} is identical on T and brings every edge of K to an edge. Then the desired retraction γ of K onto H is devised by successively applying transformations ψ_{ij}, as follows.

At the first step, set $W_1 := V(K)$ and choose a pair i, j such that there is a point $z \in W_1$ with $(z_i, z_j) \notin T_{ij}$. Set $\alpha_1 := \psi_{ij}$ and reduce W_1 to $W_2 := \alpha_1(W_1)$. Note that α decreases at least one distance, namely, for $u = \alpha_1(z)$, we have $\alpha_1(u) = u$, so $d^K(zu) > d^K(\alpha_1(z)\alpha_1(u)) = 0$. Similarly, at each step q, we choose i', j' with $(v_{i'}, v_{j'}) \notin T_{i'j'}$ for some $v \in W_q$, set $\alpha_q := \psi_{i'j'}$ and reduce W_q to $W_{q+1} := \alpha_q(W_q)$, and so on. Since each transformation is non-expansive and brings some pair of points of the current set W to closer points, the process is finite. It terminates when, after N steps, for any $z \in W_{N+1}$, each pair (z_i, z_j) is already in T_{ij}. Let $\gamma = \alpha_N \alpha_{N-1} \dots \alpha_1$. Then γ is identical on T, brings every edge to an edge and maps $V(K)$ to W_{N+1}. To conclude that γ is a retraction of K onto H, we have to show that $W_{N+1} = T$.

Statement 3.4 *Let z be a point in $V(K)$ such that $(z_i, z_j) \in T_{ij}$ for all $0 \le i < j \le k$. Then z is in H.*

Proof. For each $p = 1, \dots, k$, the set $B_p := \{t \in T : t_p = z_p\}$ is convex in H (but not necessarily in K!). Indeed, if $u, v \in B_p$ and P is a shortest u–v path in H, then P is shortest in K (since H is isometric). Therefore, $u_p = v_p = z_p$ implies $w_p = z_p$ for any node w on P, whence $w \in B_p$.

We know that the family of convex sets of a modular graph has the Helly property. The inclusion $(z_i, z_j) \in T_{ij}$ means that the sets B_i and B_j meet. Therefore, $B_1, \dots, B_k$ have a common element $z' \in T$. Clearly $z' = z$. ∎

Thus, it suffices to prove Theorem 1.4 for $k = 2$. The desired retraction will be constructed in the next section.

Remark 2. The above arguments prompt a method to solve (1.1) with μ as in Theorem 1.3 in which each particular problem concerning μ_i is solved only once (so the method looks more efficient than that described after the proof of Statement 3.1). More precisely, given V, c, find an optimal 0-extension m_i for each $i = 1, \dots, k$. This gives the family $\mathcal{X}$ of sets $X_i(z_i)$ as in (3.9), and we can select, in polynomial time, the set $\mathcal{V}$ consisting of all points $z \in K(V)$ with

Figure 3: (a) $H_{ij} \simeq K_2 \times K_2$ (b) $H_{ij} \simeq P$

$X_z \neq \emptyset$. Starting with $\mathcal{V}_1 = \mathcal{V}$, at each, qth, iteration, we examine the current set $\mathcal{V}_q$ to find $z \in \mathcal{V}$ with $(z_i, z_j) \notin T_{ij}$ for some i, j. If such a z exists and is chosen, we set $\alpha_q := \psi_{ij}$, reduce $\mathcal{V}_q$ to $\mathcal{V}_{q+1} := \alpha_q(\mathcal{V}_q)$ (which changes $\mathcal{X}$) and continue the process. Otherwise $\mathcal{V}_q = T$, by Statement 3.4, and the partition $\{Y_t : t \in T\}$ of V into the corresponding 0-distance sets induces an optimal 0-extension for V, c, d^H (and therefore, for V, c, μ, by Statement 3.1), where Y_t is the union of sets X_z for $z \in \mathcal{V}$ such that $\alpha_{q-1} \ldots \alpha_1(z) = t$. Since each transformation moves some point of the current set $\mathcal{V}$ closer to T, the number of iterations is $O(|T|^2|V|)$.

Remark 3. The above transformation of $\mathcal{X}$ induced by the retraction γ_{ij} can be thought of as an analogue of the cut uncrossing operation for median metrics (reviewed in Section 2), thus justifying the term "uncrossing" used in a more general context in this paper. Recall that each orbit graph of a median graph H is K_2, and therefore, each "two-orbit graph" H_{ij} is isomorphic either to $K_2 \times K_2$ or to the path $P = xyz$ of length two, as drawn in Fig. 3. When $H_{ij} \simeq P$, the (unique) retraction $\gamma = \gamma_{ij}$ brings the point (x, z) of $H_i \times H_j$ not in H_{ij} to y. This retraction is just behind the uncrossing operation on the corresponding cuts in that method.

4 Retraction

In this and next sections we prove Theorem 1.4 with $k = 2$, using notation, conventions and results from Sections 2 and 3. One may assume $K \neq H$. We will essentially use the condition in the theorem that H includes a subgraph ("row-layer") of the form $H_1 \times s_2$ and a subgraph ("column-layer") of the form $s_1 \times H_2$ for some $s_1 \in T_1$ and $s_2 \in T_2$, i.e.,

(4.1) for any $u \in T_1$ and $v \in T_2$, $(u, s_2) \in T$ and $(s_1, v) \in T$.

We fix such s_1, s_2 and call the node $s = (s_1, s_2)$ the *origin* of K.

In the proof below we everywhere admit that H_1, H_2 are arbitrary modular graphs until (4.9) where the assumption that H_1, H_2 are frames is essential. We abbreviate d^K, d^{H_1}, d^{H_2} as d, d_1, d_2, respectively. The *interval* $\{v \in V(K) : d(xv) + d(vy) = d(xy)\}$ of nodes (points) x, y of K is denoted by $I(x, y) = I(y, x)$. We denote by $J(x)$ and $r(x)$ the interval $I(x, s)$ and the distance $d(xs)$, called the *principal* interval and the *rank* of x, respectively. $M(x, y, z)$ denotes the set of medians of points $x, y, z \in V(K)$. For $i = 1, 2$, $I_i(x_i, y_i)$, $J_i(x_i)$, $r_i(x_i)$, and $M_i(x_i, y_i, z_i)$ stand for the corresponding objects concerning the graph H_i. Then $I(x, y) = I_1(x_1, y_1) \times I_2(x_2, y_2)$, $J(x) =$

$J_1(x_1) \times J_2(x_2)$, $r(x) = r_1(x_1) + r_2(x_2)$, and $M(x,y,z) = M_1(x_1,y_1,z_1) \times M_2(x_2,y_2,z_2)$ (as being immediate consequences from the equality $d(uv) = d_1(u_1v_1) + d_2(u_2v_2)$ for any $u, v \in V(K)$). The latter correspondence between medians in K, H_1, H_2 implies the following elementary property, which will be often used later on:

(4.2) for $x,y,z \in V(K)$ and $i \in \{1,2\}$, if $z_i \in I_i(x_i,y_i)$, then $z_i = v_i$ for each median $v \in M(x,y,z)$; in particular, $x_i = z_i$ implies $v_i = x_i$.

The modularity of H implies that

(4.3) for each $u \in T_1$, the set $Z(u) := \{v \in T_2 : (u,v) \in T\}$ is convex in H_2, and similarly for each $v \in T_2$, the set $\{u \in T_1 : (u,v) \in T\}$ is convex in H_1

(cf. the proof of Statement 3.4). Indeed, for $v, w \in Z(u)$ and $v' \in I_2(v,w)$, consider the nodes $x = (u,v)$, $y = (u,w)$ and $z = (s_1, v')$ of H (where z is in T by (4.1)). These nodes have a median q in H. Then $q_1 = u$ and $q_2 = v'$ (cf. (4.2)). Hence, $v' \in Z(u)$. It follows from (4.3) that

$$J(t) \subseteq T \qquad \text{for all } \; t \in T. \tag{4.4}$$

(However, the whole set T is not convex in K unless $H = K$.)

The mapping (retraction) γ that we wish to construct will be some kind of reflection of points in $V(K) - T$ with respect to their closest points in H. Consider a point $x \in V(K)$. Define the *excess* Δ^x to be the distance $d(x,T)$ from x to T, i.e., $\Delta^x = \min\{d(xt) : t \in T\}$, and define $N(x)$ to be the set of points $t \in T$ with $d(xt) = \Delta^x$. In particular, $\Delta^x \le r_i(x_i)$ for $i = 1,2$ (since $(x_1,s_2),(s_1,x_2) \in T$), and $\Delta^x = 0$ if and only if $x \in T$.

Statement 4.1 $N(x) \subseteq J(x)$.

Proof. Let $t \in N(x)$. The points $x' = (x_1,s_2)$, $x'' = (s_1,x_2)$ and t are in T, so they have a median q in T as well. Then $q_1 \in M_1(x_1,s_1,t_1)$ and $q_2 \in M_2(s_2,x_2,t_2)$. Therefore, q_1 belongs to both $J_1(x_1)$ and $I_1(x_1,t_1)$, and q_2 belongs to both $J_2(x_2)$ and $I_2(x_2,t_2)$, which means that $q \in J(x)$ and $q \in I(x,t)$. Now $d(xq) \ge \Delta^x = d(xt)$ implies $q = t$. ■

By this statement, the rank $r(t)$ is equal to the same number $r(x) - \Delta^x$ for all $t \in N(x)$. Note that for any $x, y \in V(K)$, $|\Delta^x - \Delta^y| = |d(x,T) - d(y,T)| \le d(xy)$. Therefore,

$$|\Delta^x - \Delta^y| \le 1 \qquad \text{for each edge } \; xy \in E(K). \tag{4.5}$$

We partition $E(K)$ into the sets $E_1 = \{xy : x_2 = y_2\}$ and $E_2 = \{xy : x_1 = y_1\}$, and for $i = 1,2$, define

$$E_i^= = \{xy \in E_i : \Delta^x = \Delta^y\} \quad \text{and} \quad E_i^{\neq} = E_i - E_i^=. \tag{4.6}$$

The desired retraction is devised by use of certain 0-extensions of metrics d_1 and d_2. First we introduce the auxiliary graphs $G_1 = (\mathcal{V}_1, \mathcal{E}_1)$ and $G_2 = (\mathcal{V}_2, \mathcal{E}_2)$, as follows. For $i = 1,2$, let $\mathcal{A}_i$ be the set of pairs $tt_i = \{t, t_i\}$ for $t \in T$, and $\mathcal{B}_i$ the set of pairs $xs_i = \{x, s_i\}$ for $x \in V(K)$. Then

G_i is the (disjoint) union of the graphs H_i and K to which the pairs from $\mathcal{A}_i \cup \mathcal{B}_i$ are added as edges, i.e.,

$$\mathcal{V}_i = T_i \cup V(K) \quad \text{and} \quad \mathcal{E}_i = U_i \cup E(K) \cup \mathcal{A}_i \cup \mathcal{B}_i.$$

The edges e of G are endowed with the lengths $\delta_i(e)$ defined by

$$\begin{aligned} \delta_i(e) &= 1 \quad \text{for} \quad e \in U_i \cup E_i^{=} \cup E_{3-i}^{\neq}, \\ &= 0 \quad \text{for} \quad e \in E_i^{\neq} \cup E_{3-i}^{=} \cup \mathcal{A}_i, \\ &= r_i(x_i) - \Delta^x \quad \text{for} \quad e = xs_i \in \mathcal{B}_i. \end{aligned} \tag{4.7}$$

We say that a semimetric m on a set V is *cyclically even* if $m(xy) + m(yz) + m(zx)$ is an even integer for all $x, y, z \in V$ (equivalently: the m-length of any cycle on V is even). All values of such an m are integers since $m(xy) + m(yx) + m(xx) = 2m(xy) \in 2\mathbf{Z}$.

Lemma 4.2 *For $i = 1, 2$, define $m_i = d^{G_i, \delta_i}$. Then:* (i) *m_i is an extension of d_i to $\mathcal{V}_i$, and* (ii) *m_i is cyclically even and coincides with δ_i on $\mathcal{E}_i$.*

This lemma (the keystone in our arguments) will be proved later, and now we explain how it help us to construct the desired mapping γ. We apply some results from [12] and [10].

More precisely, for a metric μ' on a set T', an extension m' of μ' to $V \subseteq T'$ is called *tight* if there exists no $m'' \in \mathcal{E}(\mu', V) - \{m'\}$ such that $m'' \leq m'$; equivalently: m' has no *loose* pair x, y, i.e, for any $x, y \in V$, the path (u, x, y, v) on V is m'-shortest for at least one pair $u, v \in T'$. It is shown in [12, Sec.5] that for any cyclically even metric μ',

(4.8) if $m \in \text{Ext}(\mu', V)$ is cyclically even, then there exists $m' \in \text{Ext}(\mu', V)$ such that m' is cyclically even and tight, $m'(e) \leq m(e)$ for all $e \in E_V$, and $m'(e) = m(e)$ whenever $m(e) \leq 1$.

(Such an m' is constructed by the following process. If there is no loose pair $x, y \in V$ with $m(xy) \geq 2$, then one easily shows that there is no loose pair at all, i.e., m is already tight. Otherwise choose such a pair x, y, and let $m' := d^{K_V, \ell}$, where $\ell(xy) := m(xy) - 2$ and $\ell(e) := m(e)$ for $e \in E_V - \{xy\}$. Then m' is a cyclically even extension of μ'. Update $m := m'$ and iterate.)

Next, the proof of the "if" part of Theorem 1.1 in [10] relies of an explicit construction of the so-called tight span of a frame, which in turn is based on the following result (Claim 5 in Section 4 there):

(4.9) if $H' = (T', U')$ is a frame and m is a tight extension of $d^{H'}$ to $V \supseteq T'$, then each point $x \in V$ satisfies at least one of the following:

(i) $m(xt) = 0$ for some node $t \in T'$;

(ii) $m(ux) + m(xv) = 1$ for some edge $uv \in U'$;

(iii) $m(v_0x) + m(xv_2) = m(v_1x) + m(xv_3) = 2$ for some 4-circuit $C = v_0v_1v_2v_3v_0$ of H'.

Using (4.8) and (4.9), we argue as follows. For $i = 1, 2$, let m_i be as in Lemma 4.2, and let $m'_i \le m_i$ be a cyclically even tight extension of d_i as in (4.8). Then

$$m'_i(e) = \delta_i(e) \text{ for } e \in \mathcal{E}_i - \mathcal{B}_i, \quad \text{and} \quad m'_i(e) \le \delta_i(e) \text{ for } e \in \mathcal{B}_i. \tag{4.10}$$

Moreover, in view of (4.9), for each $x \in \mathcal{V}_i$, there exists $t \in T_i$ with $m'_i(tx) = 0$. This is immediate in cases (i) and (ii) of (4.9). And if we are in case (iii) (with $m = m'_i$) and if $C = v_0v_1v_2v_3v_0$ is the corresponding 4-circuit for x, then $\alpha_j := m'_i(v_jx) > 0$ for $j = 0, 1, 2, 3$ would imply $\alpha_j = 1$ for each j. Then $m'_i(v_0v_1) + \alpha_0 + \alpha_1 = 1 + 1 + 1 = 3$, contrary to the fact that m'_i is cyclically even. Thus, m'_i is a 0-extension of d_i to $\mathcal{V}_i$.

Now for $x \in V(K)$, define $\gamma(x)$ to be the point $(\gamma_1(x), \gamma_2(x))$, where $\gamma_i(x)$ is the node $v \in T_i$ with $m'_i(xv) = 0$.

Statement 4.3 *γ is the retraction of K onto H.*

Proof. For each $t \in T$, $m'_i(tt_i) = 0$ (since δ_i is zero on $\mathcal{A}_i$, by (4.7)), so γ is identical on T.

To see $\gamma(V(K)) \subseteq T$, consider $x \in V(K)$, and let $x' = \gamma(x)$ and $t \in N(x)$. Let $P = z^0z^1 \dots z^k$ ($k = \Delta^x$) be a shortest t–x path in K. Then for $j = 0, \dots, k-1$, one has $t \in N(z^j)$ and $\Delta_j := \Delta^{z^j} = j$, whence $\Delta_{j+1} - \Delta_j = 1$ and $z^jz^{j+1} \in E_1^{\neq} \cup E_2^{\neq}$, cf. (4.6). This implies $\delta_1(P) = d_2(t_2x_2)$ and $\delta_2 = d_1(t_1x_1)$, by the definition of δ_i on $E(K)$. Therefore,

$$d_1(x'_1t_1) = m'_1(xt) \le \delta_1(P) = d_2(t_2x_2) = \Delta^x - d_1(t_1x_1). \tag{4.11}$$

Since $\delta_1(s_1x) = r_1(x_1) - \Delta^x$ (by (4.7)) and $r_1(x_1) = r_1(t_1) + d_1(t_1x_1)$ (by Statement 4.1),

$$d_1(s_1x'_1) = m'_1(s_1x) \le \delta_1(s_1x) = r_1(x_1) - \Delta^x = r_1(t_1) + d_1(t_1x_1) - \Delta^x. \tag{4.12}$$

Comparing (4.11) and (4.12), we obtain $d_1(s_1x'_1) + d_1(x'_1t_1) \le r_1(t_1)$, whence $x'_1 \in J_1(t_1)$. Similarly, $x'_2 \in J_2(t_2)$. So $x' \in J(t)$, yielding $x' \in T$, by (4.4).

Finally, consider an edge $e = xy \in E(K)$, and let $x' = \gamma(x)$ and $y' = \gamma(y)$. We have $\delta_1(e) + \delta_2(e) = 1$, by (4.7). Also $m'_i(e) = \delta_i(e)$, $i = 1, 2$, by (4.10). Hence,

$$d(x'y') = d_1(x'_1y'_1) + d_2(x'_2y'_2) = m'_1(e) + m'_2(e) = \delta_1(e) + \delta_2(e) = 1,$$

i.e., $x'y'$ is an edge of K, as required. ∎

It remains to prove Lemma 4.2.

5 Proof of Lemma 4.2

We may prove this lemma for $i = 1$. First we explain that δ_1 is cyclically even, i.e., the δ_1-length of any cycle in G_1 is even.

For any 4-circuit $C = x^0x^1x^2x^3x^0$ in K, an edge of C belongs to E_1 if and only if the opposite edge does. Also, letting $\eta_j := \Delta^{x^{j+1}} - \Delta^{x^j}$, the numbers η_0, η_2 have the same parity if and only if

η_1, η_3 do so. From these properties and the definition of δ_i one can deduce that the δ_1-length of C is even. Then δ_1 is cyclically even within K, because K is modular and, therefore, the 4-circuits form a basis in the space of cycles of K over $\mathbf{Z}_2$. (Indeed, any cycle of length $q \geq 6$ in a modular graph can be represented as the modulo two sum of three cycles with length less than q each.) Next, using the fact that δ_1 takes value one on $U_1 \cup (E_1 \cap U)$ and zero on $(E_2 \cap U) \cup \mathcal{A}_1$, one can see that the δ_1-length of any cycle with all edges in $U_1 \cup U \cup \mathcal{A}_1$ is even. Finally, for an edge $e = xs_1 \in \mathcal{B}_1$, choose $t \in N(x)$ and a shortest t–x path L in K. Then $\delta_1(L) = d_2(t_2x_2)$. Concatenating L with the edge e, the edge t_1t in $\mathcal{A}_1$ and a shortest s_1–t_1 path R in H_1, we obtain a cycle whose δ_1-length is equal to

$$\delta_1(L) + \delta_1(e) + \delta_1(R) + \delta_1(t_1t) = d_2(t_2x_2) + (r_1(x_1) - \Delta^x) + r_1(t_1) + 0 = 2r_1(t_1).$$

Summing up the above observations, one can conclude that δ_1 is cyclically even within the entire set $\mathcal{E}_1$. Then m_1 is cyclically even as well.

Next we prove that m_1 is an extension of d_1. The main part of this proof is to show the following property:

(5.1) for any path $P = x^0x^1 \dots x^k$ in K with $x^0 \in T$, there exists a path $L = z^0z^1 \dots z^\alpha$ with $z^0 = x^0$ and $z^\alpha = x^k$ and a number $0 \leq \beta \leq \alpha$ such that $z^0, \dots, z^\beta \in T$, that $r(z^\beta) < r(z^{\beta+1}) < \dots < r(z^\alpha)$, and that $\delta_1(L) \leq \delta_1(P)$.

The proof of (5.1) includes Claims 1–3 below. Recall that any edge $xy \in E(K)$ satisfies $|r(x) - r(y)| = 1$ (since K is bipartite), and if $x \in T$ and $r(x) > r(y)$, then $y \in T$ (by (4.4)). In particular, L as in (5.1) entirely lies in H if $x^k \in T$. To show (5.1), it suffices to consider the case when P is simple, $k \geq 2$, and all intermediate nodes of P are not in T (for if $x^i \in T$ for some $0 < i < k$, we can split P into two paths $P' = x^0 \dots x^i$ and $P'' = x^i \dots x^k$ and prove (5.1) for each of P', P'' independently). For $i = 0, \dots, k$, let $r(i) := r(x^i)$. An intermediate node x^i of P is called a *peak* if $r(i) > r(i-1) = r(i+1)$. The set of peaks is denoted by $F = F(P)$. We prove (5.1) by induction on

$$\omega(P) = \sum (4^{r(i)} : x^i \in F(P)).$$

If $F = \emptyset$, then $r(0) < r(1) < \dots < r(k)$ (as $r(0) > r(1)$ would imply $x^1 \in T$), i.e., P is just the desired path L. So assume $F \neq \emptyset$. Let x^p be the first peak in P, and let x, y, z stand for x^{p-1}, x^p, x^{p+1}, respectively. Choose a median y' for x, z, s in K. Since $r(x) = r(z)$ and $d(xz) = 2$, both $xy', y'z$ are edges of K and $r(y') < r(x) < r(y)$. Replace y by y' in P, forming the path $P' = x^0 \dots x^{p-1}y'x^{p+1} \dots x^k$; we say that P' is obtained by *cutting off* the peak y. Since $4^{r(p)} > 2 \cdot 4^{r(p)-1} = 4^{r(p-1)} + 4^{r(p+1)}$, we have $\omega(P') < \omega(P)$. Also $\delta_1(P) - \delta_1(P')$ is equal to

$$\rho := \rho(x, y, z, y') := \delta_1(xy) + \delta_1(yz) - \delta_1(xy') - \delta_1(y'z).$$

Therefore, if $\rho \geq 0$ occurs, we can immediately apply induction. Let $\overline{\Delta} := \Delta^y$.

Claim 1 *A median y' for x, z, s can be chosen so that $\rho(x, y, z, y') < 0$ is possible only if both edges $e = xy, e' = yz$ are in E_2, $\Delta^x = \Delta^z = \overline{\Delta}$, and $\Delta^{y'} = \overline{\Delta} - 1$.*

Proof. Since the δ_1-length of the 4-circuit $C = xyzy'x$ is even, $\rho < 0$ implies

$$\delta_1(e) = \delta_1(e') = 0 \qquad \text{and} \qquad \delta_1(xy') = \delta_1(y'z) = 1. \tag{5.2}$$

This is impossible when $e \in E_1$ and $e' \in E_2$ (or $e \in E_2$ and $e' \in E_1$). Indeed, in this case we would have $\Delta^x = \overline{\Delta} - 1$ and $\Delta^z = \overline{\Delta}$, by (4.7). Then $xy' \in E_2$ and $\delta_1(xy') = 1$ imply $\Delta^{y'} = \Delta^x - 1 = \overline{\Delta} - 2$, while $y'z \in E_1$ and $\delta_1(y'z) = 1$ imply $\Delta^{y'} = \Delta^z = \overline{\Delta}$; a contradiction.

If $e, e' \in E_2$, then $xy', y'z \in E_2$. So (5.2) yields $\Delta^x = \Delta^z = \overline{\Delta}$ and $\Delta^{y'} = \Delta^x - 1 = \overline{\Delta} - 1$, as required.

Now, suppose $e, e' \in E_1$ and $\delta_1(e) = \delta_1(e') = 0$. Choose $u \in N(x)$ and $v \in N(z)$. We have $\Delta^x = \Delta^z = \overline{\Delta} - 1$, whence $u, v \in N(y)$. Choose in T a median q for $u, v, (y_1, s_2)$ and a median w for $u, v, (s_1, y_2)$. We assert that $q, w \in N(y)$. Indeed,

$$q_1 \in M_1(u_1, v_1, y_1), \quad w_1 \in M_1(u_1, v_1, s_1), \quad q_2 \in M_1(u_2, v_2, s_2), \quad w_2 \in M_1(u_2, v_2, y_2).$$

In particular, $q_1, w_1 \in I_1(u_1, v_1)$. Also $u_1, v_1 \in I_1(q_1, w_1)$ (in view of $u_1, v_1 \in I_1(y_1, s_1)$, by Statement 4.1). These relations imply $d_1(u_1q_1) = d_1(v_1w_1) := a$. Similarly, $d_2(u_2q_2) = d_2(v_2w_2) := a'$. Then $d(yu) = \overline{\Delta} \le d(yq) = d(yu) - a + a'$ and $d(yv) \le d(yw) = d(yv) + a - a'$. This is possible only if $a = a'$, yielding $d(yq) = d(yw) = \overline{\Delta}$, as required.

Assume y' is chosen to be a median for x, z, w. Then y' is a median for x, z, s as well, taking into account that $x_2 = z_2$ and the paths (x_1, u_1, w_1, s_1) and (z_1, v_1, w_1, s_1) on T_1 are d_1-shortest. Now $d(y'w) = d(xw) - 1$ implies $\Delta^{y'} < \Delta^x$. Hence, $\delta_1(xy') = \delta_1(y'z) = 0$ and $\rho = 0$. ∎

Arguing as in the above proof, one can see that for any $x' \in V(K)$, there are elements $t, t' \in N(x')$ such that $r_1(t_1) \le r_1(t'_1)$ (and $r_2(t_2) \ge r_2(t')$) and $N(x') \subseteq I(t, t')$. We denote t by $t(x')$ and refer to it as the *minimal* element of $N(x')$ (with respect to the rank in H_1).

Remark 4. For $i = 1, 2$ and $f, g \in N(x')$, denote $f_i \prec_i g_i$ if $f_i \in J_i(g_i)$. Then $\prec_i$ is the partial order on $N_i = \{w_i : w \in N(x')\}$ with unique minimal and maximal elements. Moreover, the correspondence $w_1 \to w_2$ establishes the isomorphism between $(N_1, \prec_1)$ and $(N_2, \prec_2^{-1})$ (where $\prec^{-1}$ is the reverse to $\prec$). One can show that if none of H_1, H_2 containes $K_{3,3}^-$ as an induced subgraph (see Fig. 1b), then $(N_i, \prec_i)$ is a *modular lattice*, i.e., (i) any $u, v \in N_i$ have unique lower and upper bounds, denoted by $u \wedge v$ and $u \vee v$, respectively; (ii) for each $u \in N_i$, all maximal chains to u from the minimal element have the same length $\rho(u)$, and (iii) each pair u, v satisfies the modular equality $\rho(u) + \rho(v) = \rho(u \wedge v) + \rho(u \vee v)$. We, however, do not need these properties in further arguments.

In light of Claim 1, we may assume that $\rho < 0$ and $e, e' \in E_2$. Consider the minimal element $t(y) = (t_1(y), t_2(y))$ in $N(y)$. Suppose $t_1(y) \ne y_1$. Then there is a node w of K adjacent to y such that $w_1 \in I_1(y_1, t_1(y))$ and $w_2 = y_2$. We have $yw \in E_1$, $r(w) = r(y) - 1$ and $t(y) \in N(w)$. Then $\Delta^w < \overline{\Delta}$ and $\delta_1(yw) = 0$. Transform P into the (non-simple) path $P' = x^0 \ldots x^{p-2}xywyzx^{p+2} \ldots x^k$ and then cut off both copies of y (which are peaks of P'). This results in a path P'' of the form $x^0 \ldots x^{p-2}xy'wy''zx^{p+2} \ldots x^k$; clearly x, w, z are peaks of P''.

Since $yw \in E_1$, y' and y'' can be chosen so that $\rho(x, y, w, y') \geq 0$ and $\rho(w, y, z, y'') \geq 0$, by Claim 1. Therefore, $\delta_1(P'') \leq \delta_1(P') = \delta_1(P)$. Also $4^{r(y)} > 3 \cdot 4^{r(y)-1} = 4^{r(x)} + 4^{r(w)} + 4^{r(z)}$, yielding $\omega(P'') < \omega(P)$. So we can apply induction.

It remains to consider the case when $t_1(y) = y_1$. Then $t(y)$ is the unique element of $N(y)$. We will use the following property.

Claim 2. *Let $\overline{x}\overline{y} \in E_2$ satisfy $r(\overline{x}) < r(\overline{y})$, let $N(\overline{y})$ consist of a single element u, and let $u_1 = \overline{y}_1$. Then $N(\overline{x})$ consists of a single element v, and $v_1 = \overline{y}_1$. Moreover, $u = v$ if $\Delta^{\overline{x}} < \Delta^{\overline{y}}$, and u and v are adjacent if $\Delta^{\overline{x}} = \Delta^{\overline{y}}$.*

Proof. If $\Delta^{\overline{x}} < \Delta^{\overline{y}}$, then $N(\overline{x}) \subseteq N(\overline{y})$, whence $N(\overline{x}) = \{u\}$. So assume $\Delta^{\overline{x}} = \Delta^{\overline{y}}$, and let $v \in N(\overline{x})$. Choose $q \in M(u, v, (\overline{y}_1, s_2)) \cap T$ and $w \in M(u, v, (s_1, \overline{y}_2)) \cap T$. We have $q_2, w_2 \in I_2(u_2, v_2)$ and $u_2 \in I_2(q_2, w_2)$ (in view of $u_2 \in I_2(\overline{y}_2, s_2)$). Note that the path $(\overline{y}_2, \overline{x}_2, v_2, s_2)$ on T_2 is d_2-shortest (since $r(\overline{x}) < r(\overline{y})$ and $\overline{x}_1 = \overline{y}_1$ imply $\overline{x}_2 \in I_2(\overline{y}_2, s_2)$). This yields $v_2 \in I_2(q_2, w_2)$, and we can conclude that $d_2(u_2w_2) = d_2(v_2q_2) =: a'$.

Next, $q_1 \in M_1(u_1, v_1, \overline{y}_1)$ and $u_1 = \overline{y}_1$ imply $q_1 = \overline{y}_1$, while $w_1 \in M_1(u_1, v_1, s_1)$, $v_1 \in I_1(\overline{x}_1, s_1)$ and $\overline{x}_1 = \overline{y}_1 = \overline{u}_1$ imply $w_1 = v_1$. Let $a := d_1(\overline{y}_1v_1)$. Then $d(\overline{x}v) \leq d(\overline{x}q) = d(\overline{x}v) - a + a'$ and $d(\overline{y}u) \leq d(\overline{y}w) = d(\overline{y}u) + a - a'$, whence $a = a'$, $q \in N(\overline{x})$ and $w \in N(\overline{y})$. Since $|N(\overline{y})| = 1$, we have $w = u$. This implies $a = 0$ and $q = v$, yielding $v_1 = q_1 = \overline{y}_1$. So $v_1 = \overline{y}_1$, regardless of the choice of v in $N(\overline{x})$. This is possible only if $N(\overline{x})$ consists of a single element (for if $v, v' \in N(\overline{x})$ and $v \neq v'$, then a median f for $v, v', (s_1, \overline{x}_2)$ in T satisfies $f_1 = \overline{y}_1$ and $d_2(\overline{x}_2f_2) < d_2(\overline{x}_2v_2)$, whence $d(\overline{x}f) < \Delta^{\overline{x}}$).

Finally, to see that u_2 and v_2 are adjacent, take in T a median h for $u, v, (s_1, \overline{x}_2)$. Then $d(\overline{x}h) \geq d(\overline{x}v)$, $h_2 \in I_2(\overline{x}_2, v_2)$ and $h_1 = \overline{y}_1$, implying $h = v$. So $v_2 \in I_2(\overline{x}_2, u_2)$. Also $d_2(\overline{y}_2u_2) = \Delta^{\overline{y}} = \Delta^{\overline{x}} = d_2(\overline{x}_2v_2)$ and $u_2 \in I_2(\overline{y}_2, v_2)$ (since $u_2 \in I_2(\overline{y}_2, s_2)$ and $v_2 = q_2 \in I_2(u_2, s_2)$). Now $d_2(\overline{x}_2\overline{y}_2) = 1$ implies $d_2(u_2v_2) = 1$, as required. ■

For $i = 0, \dots, p$, define P_i to be the subpath $x^i \dots x^p$ of P. Let P_j be the maximal subpath with all edges in E_2 (i.e., j is minimum subject to $x_1^j = \dots = x_1^p$). Since $r(j) < r(j+1) < \dots < r(p)$, we can repeatedly apply Claim 1 to the edges of P_j, starting with $x^{p-1}x^p$, and conclude that $N(x^i)$ is a singleton $\{u^i\}$ with $u_1^i = y_1$ for each $i = j, \dots, p$. Also $u^i = u^{i+1}$ if $\Delta_i < \Delta_{i+1}$, and $u^iu^{i+1} \in E_2$ if $\Delta_i = \Delta_{i+1}$, where Δ_q stands for Δ^{x^q}. Consider two possible cases.

Case 1: $j \geq 1$. By the maximality of P_j, $x^{j-1}x^j \in E_1$. Let $b := x_1^{j-1}$. For $i = j, \dots, p$, define z^i and v^i to be the points with $z_1^i = v_1^i = b$, $z_2^i = x_2^i$ and $v_2^i = u_2^i$, i.e., z^i and v^i are obtained by shifting the points x^i and u^i, respectively, along the edge y_1b of H_1. In particular, $z^j = x^{j-1}$. Denote Δ^{z^i} by Δ'_i.

Claim 3. *$\Delta'_i = \Delta_i$ and $v^i \in N(z^i)$ for each $i = j, \dots, p$.*

Proof. Since $r(j-1) < r(j)$ and $x_2^{j-1} = x_2^j$, $r_1(b) < r_1(x^j)$. Therefore, $u^i \in T$ implies $v^i \in T$, and we have $\Delta'_i \leq d_2(z^iv^i) = d_2(x^iu^i) = \Delta_i$. Suppose $\Delta'_i < \Delta_i$. Then $N(z^i) \subseteq N(x^i)$, whence $N(z^i) = \{u^i\}$. But $d(z^iu^i) = d_1(by_1) + d_2(x_2^iu_2^i) = 1 + d(z^iv^i)$; a contradiction. Thus, $\Delta'_i = \Delta_i$ and $v^i \in N(z^i)$. ■

Consider the x^{j-1}–y paths P_{j-1} and $R = z^j \dots z^p x^p$ in K. From Claim 3 it follows that $\delta_1(z^i z^{i+1}) = \delta_1(x^i x^{i+1})$ for $i = j, \dots, p-1$, and that $\delta_1(x^{j-1}x^j) = \delta_1(z^p x^p)$. Therefore, $\delta_1(P_{j-1}) = \delta_1(R)$. Replace in P the part P_{j-1} by R, forming the path $P' = x^0 \dots x^{j-1} z^j \dots z^p x^p \dots x^k$. Clearly $y = x^p$ is the first peak of P'. Cut off y in P' by replacing y by a median y'' for z^p, z, s; let P'' be the resulting path. Since $z^p y \in E_1$ and $yz \in E_2$, one has $\rho(z^p, y, z, y'') \geq 0$, by Claim 1. Therefore, $\delta_1(P'') \leq \delta_1(P') = \delta_1(P)$, and (5.1) follows by induction because z^p and z are the first and second peaks of P'' and $4^{r(y)} > 4^{r(z^p)} + 4^{r(z)}$.

Case 2: $j = 0$. Then $x^0 = u^0$. By Claim 2 applied to the edge zy, $N(z)$ is a singleton $\{\hat{u}\}$ with $\hat{u}_1 = y_1$. As before, let $y' \in M(x, z, s) \cap T$; then $y'_1 = y_1$ and $N(y')$ is a singleton $\{v\}$ (by Claim 2 applied to the edge $y'x$). Assuming $\Delta^{y'} < \Delta^x$ (equivalently: $\rho < 0$), we have $N(y') \subseteq N(x) \cap N(z)$. Hence, $v = \hat{u} = u^{p-1}$.

Form the u^0–v path R' by deleting repeated consecutive elements in $u^0 \dots u^{p-1}$, and let $\overline{R}$ be the concatenation of R', a shortest v–y' path R'', and the edge $y'x$. Clearly the δ_1-length of each edge of R' is zero, while the δ_1-length of each edge of R'' is one. Also $\delta_1(y'x) = 1$.

Comparing $\overline{R}$ with the path $\overline{P} = x^0 \dots x^{p-1}$ and using Claim 2, one can deduce that $\overline{R} = p - 1$ (i.e., $\overline{R}$ is a shortest path in K) and that $\delta_1(\overline{R}) = \delta_1(\overline{P})$. Now let D be the concatenation of R', R'' and the edge $y'z$. Since $\delta_1(y'x) = \delta_1(y'z)$ and $\delta_1(xy) = \delta_1(yz) = 0$, we have $\delta_1(D) = \delta_1(\overline{R}) = \delta_1(P_0)$. Also $|D| = |R| = p - 1$ implies that D has no peaks. Then, replacing in P the part $x^0 \dots x^{p+1}$ by D, we obtain the path P' with $\delta_1(P') = \delta(P)$ and $\omega(P') < \omega(P)$ and can apply induction.

Thus, (5.1) is proven. In order to conclude that m_1 is an extension of d_1, it suffices to consider a path L as in (5.1) and show the following:

(5.3) (i) if $z^\alpha \in T$, then $\delta_1(L) \geq d_1(z_1^0 z_1^\alpha)$;

(ii) $\delta_1(L) + \delta_1(z^\alpha s_1) \geq r_1(z_1^0)$.

(In fact, (i) embraces the case of a path in G_1, with both ends in T_1, whose first and last edges belong to $\mathcal{A}_1$, while (ii) does the case when one of these edges is in $\mathcal{A}_1$ and the other in $\mathcal{B}_1$.) Case (i) is trivial because $z^\alpha \in T$ means that L is a path in H, and therefore, the δ_1-length of each of its edges in E_1 is equal to one. So let us prove (ii). One may assume that $r(z^0) < \dots < r(z^\alpha)$ (taking into account that $d_1(z_1^0 s_1) \leq d_1(z_1^0 z_1^\beta) + d_1(z_1^\beta z_1^\alpha)$ and $\delta_1(L) = \delta_1(L') + \delta_1(L'')$, where $L' = z^0 \dots z^\beta$ and $L'' = z^\beta \dots z^\alpha$, and assuming w.l.o.g. that L' is δ_1-shortest).

For $i = 0, \dots, \alpha$, let ℓ_i denote the δ_1-length of the path $z^0 \dots z^i$, and let ρ_i and Δ_i stand for $r_1(z_1^i)$ and Δ^{z^i}, respectively. By the definition of δ_1 on $\mathcal{B}_1$, $\delta_1(z^i s_1)$ is equal to $\rho_i - \Delta_i$. We show that

$$\ell_i + \rho - \Delta_i \geq \rho_0, \tag{5.4}$$

using induction on i. This gives the desired inequality (5.3)(ii) when $i = \alpha$. Since $\ell_0 = \Delta_0 = 0$, (5.4) holds for $i = 0$. Assume it holds for $i - 1$ $(0 < i < \alpha)$, and let $a := \ell_i - \ell_{i-1}$, $b := \rho_i - \rho_{i-1}$ and $c := \Delta_i - \Delta_{i-1}$. Then (5.4) for i follows from $a + b - c \geq 0$. To see the latter, consider four possible cases for $e = z^{i-1}z^i$, taking into account that $\Delta_i \geq \Delta_{i-1}$ since $r(z^i) > r(z^{i-1})$.

(a) Let $e \in E_1$ and $\Delta_i = \Delta_{i-1}$. Then $a + b - c = 1 + 1 + 0 = 2$.
(b) Let $e \in E_1$ and $\Delta_i > \Delta_{i-1}$. Then $a + b - c = 0 + 1 - 1 = 0$.
(c) Let $e \in E_2$ and $\Delta_i = \Delta_{i-1}$. Then $a + b - c = 0 + 0 - 0 = 0$.
(d) Let $e \in E_2$ and $\Delta_i > \Delta_{i-1}$. Then $a + b - c = 1 + 0 - 1 = 0$.

Thus, m_1 is an extension of d_1. It remains to show that $m_i(e) = \delta_i(e)$ for $i = 1, 2$ and $e \in \mathcal{E}_i$. This is obvious when $e \in U \cup U_i$ or when $\delta_i(e) = 0$. If $e = xs_i \in \mathcal{B}_i$, then $m_i(e) = \delta_i(e)$ follows from the fact that for $t \in N(x)$, the path in G_i obtained by concatenationg the edge $t_i t$, a shortest t–x path in K, and the edge xs_i is δ_i-shortest (this fact was shown at the beginning of this section). Finally, each edge $e \in E(K)$ belongs to a shortest t–t' path P in K with $t, t' \in T$. Since $\delta_1(e') + \delta_2(e') = 1$ for all edges e' of K, we have $\delta_1(P) + \delta_2(P) = |P| = d(tt') = d_1(t_1 t'_1) + d_2(t_2 t'_2)$, whence $\delta_i(P) = d_i(t_i t'_i)$, implying $m_i(e) = \delta_i(e)$.

This completes the proof of Lemma 4.2 and completes the proof of Theorems 1.4 and 1.3.

6 Intractable Cases

In this section we prove Theorem 1.6, considering a metric μ on a set T such that either μ is non-modular or μ is modular but its underlying graph $H = (T, U)$ is non-orientable. W.l.o.g., one may assume μ is integer-valued. Our method borrows the idea from [10] for the path metrics $\mu = d^H$ as in Theorem 1.5, which in turn generalizes the construction from [6] for $H = K_3$.

Given a set $V \supset T$, a function $E_V \to \mathbf{Z}_+$, nodes $s, t \in T$, and points $x, y \in V - T$, let $\tau(s, x|t, y)$ denote the minimum $c \cdot m$ among all $m \in \mathrm{Ext}^0(\mu, V)$ such that $m(xs) = m(yt) = 0$.

The core of the proof in [6] that the 3-terminal cut problem is NP-hard is the construction of a "gadget" (V, c) with specified s, t, x, y satisfying the following property:

(6.1) (i) $\tau(s, x|t, y) = \tau(s, y|t, x) = \hat{\tau}$,

(ii) $\tau(s, x|s, y) = \tau(t, x|t, y) = \hat{\tau} + \delta$ for some $\delta > 0$,

(iii) $\tau(s', x|t', y) \geq \hat{\tau} + \delta$ for all other pairs $\{s', t'\}$ in T,

where $\hat{\tau}$ stands for $\tau(V, c, \mu)$ (with $\mu = d^{K_3}$). Then the NP-hardness of the problem is easily shown by a reduction from MAX CUT.

Our aim is to construct corresponding "gadgets" satisfying (6.1) for μ as in Theorem 1.6; then the theorem will follow by a similar reduction.

First we consider the case when μ is modular but H is non-orientable, which is technically simpler. In fact, the construction and arguments in this case are similar to those for the corresponding unweighted case ($\mu = d^H$) given in [10, Sec. 6]. More precisely, since H is non-orientable, there exists a projective sequence $(e_0, e_1, \ldots, e_{k-1}, e_k = e_0)$ of edges of H yielding the "twist" (or forming the *orientation-reversing dual cycle*). That is,

(6.2) for $i = 0, \ldots, k-1$, $e_i = s_i t_i$ and $e_{i+1} = s_{i+1} t_{i+1}$ are opposite edges in the 4-circuit $C_i = s_i t_i t_{i+1} s_{i+1} s_i$, and $t_k = s_0$ (and $s_k = t_0$).

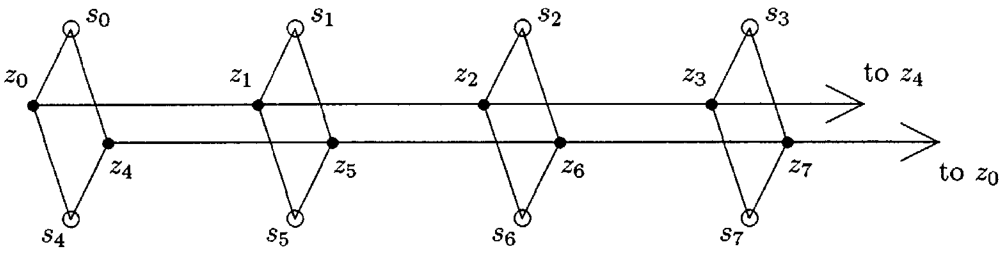

Figure 4: gadget for a non-orientable H

(One can choose such a sequence with all edges (though not necessarily the nodes) distinct, but this is not important for us.) Since μ is modular, we have by (2.2) that

(6.3) for $i = 0, \ldots, k-1$, $\mu(e_i)$ is a constant h, and $\mu(s_is_{i+1}) = \mu(t_it_{i+1}) =: f_i$.

We denote t_i by s_{i+k} and take indices modulo $2k$. The desired gadget is represented by the graph $G = (V, E)$ with the weights $c(e)$ of edges $e \in E$, where $V = T \cup \{z_0, \ldots, z_{2k-1}\}$ and for $i = 0, \ldots, 2k-1$,

(i) z_i is adjacent to both s_i and s_{i+k}, and $c(z_is_i) = c(z_is_{i+k}) = N$ for a positive integer N (specified below);

(ii) z_i and z_{i+1} are adjacent, and $c(z_iz_{i+1}) = 1$.

Figure 4 illustrates G for $k = 4$. We put $s = s_0$, $t = t_0$, $x = z_0$ and $y = z_k$, and formally extend c by zero to $E_V - E$. We assert that (6.1) holds.

Indeed, each $m \in \mathrm{Ext}^0(\mu, V)$ is associated with the mapping $\gamma : \{z_0, \ldots, z_{2k-1}\} \to T$, where $\gamma(z_i) = s_j$ if $m(z_is_j) = 0$; we say that z_i is *attached* by γ to s_j and denote m by m^γ. If $\gamma(z_i) = v$, then, letting $\epsilon := \mu(s_iv) + \mu(vs_{i+k}) - \mu(s_is_{i+k})$, the contribution to the *volume* $c \cdot m^\gamma$ due to the edges $e = z_is_i$ and $e' = z_is_{i+k}$ is equal to

$$c(e)m^\gamma(e) + c(e')m^\gamma(e') = N(m^\gamma(e) + m^\gamma(e')) = Nh + N\epsilon;$$

cf. (6.3). We have $\epsilon = 0$ if $v \in \{s_i, s_{i+k}\}$, and $\epsilon \geq 1$ otherwise. Hence, every mapping γ pretending to be optimal or nearly optimal must attach each z_i to either s_i or s_{i+k} whenever N is chosen sufficiently large (e.g., $N = 1 + 2k \max\{\mu(st) : s, t \in T\}$).

Next, if z_i is attached to s_i (resp. s_{i+k}) and z_{i+1} to s_{i+1} (resp. s_{i+1+k}), then the edge $u = z_iz_{i+1}$ contributes $c(u)m^\gamma(u) = f_i$ (cf. (6.3), letting $f_j = f_{j+k}$. On the other hand, if z_i is attached to s_i (resp. s_{i+k}) while z_{i+1} to s_{i+1+k} (resp. s_{i+1}), then the contribution becomes $h + f_i$ $(= \mu(s_it_{i+1}))$.

So we can conclude that $\hat{\tau} = 2khN + 2(f_1 + \ldots + f_k)$, and there are precisely two optimal 0-extensions, namely, m^{γ_1} and m^{γ_2}, where $\gamma_1(z_i) = s_i$ and $\gamma_2(z_i) = s_{i+k}$ for $i = 0, \ldots, 2k-1$. This gives (i) in (6.1). Furthermore, one can see that if m^γ is the least-volume 0-extension induced by γ that brings both x, y either to s or to t, then $m^\gamma(z_j z_{j+1}) = h + f_j$ for precisely two numbers $j \in \{0, \ldots, 2k-1\}$ such that $f_j = \min\{f_1, \ldots, f_k\}$. So $c \cdot m^\gamma = \hat{r} + 2h$, yielding (6.1)(ii). Finally, (iii) is ensured by the choice of N.

Thus, (1.1) with μ modular and H non-orientable is NP-hard. Moreover, it is strongly NP-hard because the number N is a constant depending only on μ.

Next we consider the case when μ is not modular. Let $\Delta(x, y, z)$ denote the value (*perimeter*) $\mu(xy) + \mu(yz) + \mu(zx)$ for $x, y, z \in T$. We fix a medianless triplet $\{s_0, s_1, s_2\}$ such that $\Delta(s_0, s_1, s_2) := \overline{\Delta}$ is minimum. By technical reasons, we put $s_{i+3} = s_i$, $i = 0, 1, 2$, and take indices modulo 6. The gadget $(G = (V, E), c)$ that we construct has a somewhat more complicated structure compared with that for the corresponding unweighted case in [10, Sec. 6]. Here

$$V = T \cup Z, \quad Z = \{z_0, \ldots, z_5\} \quad \text{and} \quad E = E_1 \cup E_2 \cup E_3.$$

For $i = 1, 2, 3$, the edges $e \in E_i$ are endowed with weights $c_i(e)$, and $c(e)$ is defined to be $N_i c_i(e)$. The factors N_1, N_2, N_3 are chosen so that $N_1 = 1$, N_2 is sufficiently large, and N_3 is sufficiently large with respect to N_2. Informally speaking, the "heavy" edges of E_3 provide that (at optimality or almost optimality) each point z_j gets into the interval $I_j := \{v \in T : \mu(s_{j-1}v) + \mu(v s_{j+1}) = \mu(s_{j-1}s_{j+1})\}$, then the "medium" edges of E_2 make z_j choose only between the endpoints s_{j-1}, s_{j+1} of I_j, and finally the "light" edges of E_1 provide the desired property (6.1).

As before, m^γ denotes the 0-extension of μ to V induced by $\gamma : Z \to T$. Define $d_i := d_{i+3} = \mu(s_{i-1}s_{i+1})$. We say that a path $P = (v_1, \ldots, v_k)$ on T is shortest if it is μ-shortest.

The set E_3 consists of the edges $e_j = z_j s_{j-1}$ and $e'_j = z_j s_{j+1}$ with $c_3(e_j) = c_3(e'_j) = 1$ for $j = 0, \ldots, 5$. Then the contribution to $c \cdot m^\gamma$ due to e_j and e'_j is $N_3 d_j$ if $\gamma(z_j) \in I_j$, and at least $N_3 d_j + N_3$ otherwise, yielding that z_j should be mapped into I_j, by the choice of N_3. The minimality of $\overline{\Delta}$ provides the following useful property.

Statement 6.1 *For any $v \in I_j$, at least one of the paths $P = (s_j, s_{j-1}, v)$ and $P' = (s_j, s_{j+1}, v)$ is shortest.*

Proof. Let for definiteness $j = 1$. Suppose P' is not shortest. Then $\mu(s_1 v) < |P'| = \mu(s_1 s_2) + \mu(s_2 v)$ and $\mu(s_0 v) = \mu(s_0 s_2) - \mu(s_2 v)$ imply $\Delta(s_1, v, s_0) < \overline{\Delta}$. So s_1, v, s_0 have a median w. If $w = s_0$, P is shortest. Otherwise we have $\Delta(s_1, w, s_2) < \Delta$ (since $\mu(s_1 w) < \mu(s_1 s_0)$ and the path (s_2, v, w, s_0) is, obviously, shortest). Then s_1, w, s_2 have a median q. It is easy to see that q is a median for s_0, s_1, s_2; a contradiction. ■

We now explain the construction of E_2 and c_2. Each $z = z_j$ $(j = 0, \ldots, 5)$ is connected to each s_i $(i = 0, 1, 2)$ by edge $u_i = z s_i$ whose weight is defined by

$$c_2(u_i) = (d_{i-1} + d_{i+1} - d_i)/(d_{i-1} d_{i+1}) =: a_i \tag{6.4}$$

(a_i is positive and does not depend on j). Suppose z is mapped by γ to some s_i, say $\gamma(z) = s_1$. Then, up to a factor of N_2, the contribution to $c \cdot m^\gamma$ from the edges u_0, u_1, u_2 (concerning z) is

$$d_2a_0 + d_0a_2 = d_2(d_1 + d_2 - d_0)/(d_1d_2) + d_0(d_1 + d_0 - d_2)/(d_0d_1) \qquad (6.5)$$
$$(d_1 + d_2 - d_0)/d_1 + (d_1 + d_0 - d_2)/d_1 = 2.$$

On the other hand, the contribution grows when z_j falls into the interior of any interval I_i.

Statement 6.2 *Let $v \in I_i - \{s_{i-1}, s_{i+1}\}$. Then $\sigma := \sum(a_i\mu(s_iv) : i = 0, 1, 2) > 2$.*

Proof. Let for definiteness $i = 0$, $\mu(s_1v) = \epsilon$ and $\mu(s_0v) = d_2 + \epsilon$ (cf. Statement 6.1). Then

$$\sigma = (d_2 + \epsilon)a_0 + \epsilon a_1 + (d_0 - \epsilon)a_2 = d_2a_0 + d_0a_2 + \epsilon(a_0 + a_1 - a_2) = 2 + \epsilon(a_0 + a_1 - a_2),$$

in view of (6.5). We observe that $a_0 + a_1 - a_2 > 0$. Indeed,

$$d_0d_1d_2(a_0 + a_1 - a_2) = (d_0d_1 + d_0d_2 - d_0^2) + (d_1d_0 + d_1d_2 - d_1^2) - (d_2d_0 + d_2d_1 - d_2^2)$$
$$= 2d_0d_1 - d_0^2 - d_1^2 + d_2^2 = d_2^2 - (d_0 - d_1)^2 > 0$$

since $d_2 > d_0 - d_1$. So $\sigma > 2$. ■

Thus, by an appropriate choice of constants N_2 and N_3, each point z_j must be mapped to either s_{j-1} or s_{j+1}. Such a mapping γ is called *feasible.* We now construct the crucial set E_1 and function c_1. The set E_1 consists of six edges $g_j = z_jz_{j+1}$, $j = 0, \ldots, 5$, forming the 6-circuit C (this is similar to the construction in [10] motivated by [6]). The essense is how to assign c_1. For $i = 0, 1, 2$, let $h_i := h_{i+3} := (d_{i-1} + d_{i+1} - d_i)/2$. These numbers would be just the distances from s_0, s_1, s_2 to their median if it existed, i.e.,

$$d_i = h_{i-1} + h_{i+1}. \qquad (6.6)$$

We define

$$c_1(z_jz_{j+1}) = c_1(z_{j+3}z_{j+4}) = h_{j-1} \qquad \text{for} \quad j = 0, 1, 2. \qquad (6.7)$$

For $\gamma : Z \to T$, let ζ^γ denotes $\sum(c_1(g_j)m^\gamma(g_j) : j = 0, \ldots, 5)$, i.e., ζ^γ is the contribution to $c \cdot m^\gamma$ from the edges of C. The analysis below will depend on the numbers

$$\rho = 2(h_0h_1 + h_1h_2 + h_2h_0) \quad \text{and} \quad \alpha = 2\min\{h_0^2, h_1^2, h_2^2\}. \qquad (6.8)$$

W.l.o.g., assume $h_0 \leq h_1, h_2$, i.e., $2h_0^2 = \alpha$. Our aim is to show that (6.1) holds if we take as s, t, x, y the elements s_0, s_2, z_1, z_4, respectively.

To show this, consider the mapping γ_1 as drawn in Fig. 5a, i.e., $\gamma_1(z_j)$ is s_{j+1} for $j = 0, 2, 4$ and s_{j-1} for $j = 1, 3, 5$. This γ_1 attaches x to s and y to t. In view of (6.6)-(6.8), we have

$$\zeta^{\gamma_1} = c_1(g_0)\mu(\gamma_1(z_0)\gamma_1(z_1)) + \ldots + c_1(g_5)\mu(\gamma_1(z_5)\gamma_1(z_0))$$
$$= h_2d_2 + h_0 \cdot 0 + h_1d_1 + h_2 \cdot 0 + h_0d_0 + h_1 \cdot 0 = h_2(h_0 + h_1) + h_1(h_0 + h_2) + h_0(h_1 + h_2) = \rho.$$

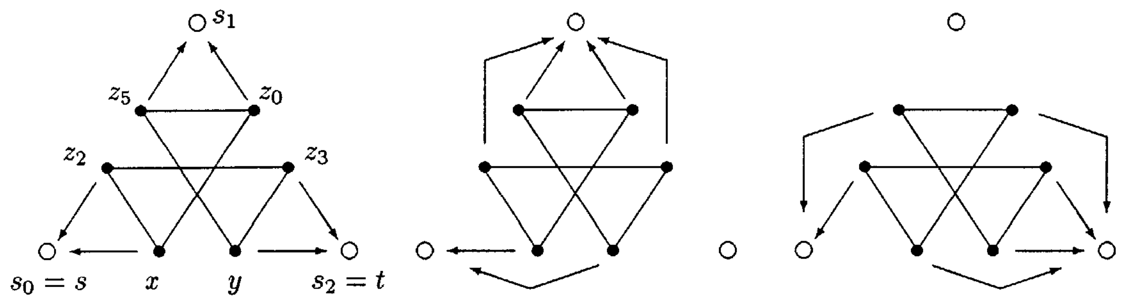

Figure 5: (a) γ_1 (b) γ_3 (c) γ_4

Similarly, $\zeta^{\gamma_2} = \rho$ for the symmetric mapping γ_2 which is defined by $\gamma_2(z_j) = \gamma_1(z_{j+3})$, attaching x to t and y to s. We shall see later that γ_1 and γ_2 are just optimal mappings for our gadget.

The mappings pretending to provide (ii) in (6.1) are γ_3 and γ_4 illustrated in Fig. 5b,c; here both x, y are mapped by γ_3 to s, and by γ_4 to t. We have

$$\begin{aligned}\zeta^{\gamma_3} &= h_2d_2 + h_0d_2 + h_1 \cdot 0 + h_2d_2 + h_0d_2 + h_1 \cdot 0 = (2h_2 + 2h_0)(h_0 + h_1)\\ &= 2h_2h_0 + 2h_2h_1 + 2h_0^2 + 2h_0h_1 = \rho + \alpha\end{aligned}$$

and

$$\begin{aligned}\zeta^{\gamma_4} &= h_2 \cdot 0 + h_0d_1 + h_1d_1 + h_2 \cdot 0 + h_0d_1 + h_1d_1 = (2h_0 + 2h_1)(h_0 + h_2)\\ &= 2h_0^2 + 2h_0h_2 + 2h_1h_0 + 2h_1h_2 = \rho + \alpha.\end{aligned}$$

Now (6.1) is implied by the following.

Statement 6.3 *Let γ be a feasible mapping different from γ_1 and γ_2. Then $\zeta^\gamma \geq \rho + \alpha$.*

Proof. By (6.6), ζ^γ is representable as a nonnegative integer combination of products h_ih_j for $0 \leq i, j \leq 5$ (including $i = j$). The contribution ζ_j to $c \cdot m^\gamma$ from a single edge $g_j = z_jz_{j+1}$ is as follows:

(6.9) (i) if $\gamma(z_j) = \gamma(z_{j+1}) = s_{j-1}$, then $\zeta_j = 0$;

(ii) if $\gamma(z_j) = s_{j+1}$ and $\gamma(z_{j+1}) = s_j$, then $\zeta_j = h_{j-1}d_{j-1} = h_{j-1}h_j + h_{j-1}h_{j+1}$;

(iii) if $\gamma(z_j) = s_{j+1}$ and $\gamma(z_{j+1}) = s_{j-1}$, then $\zeta_j = h_{j-1}d_j = h_{j-1}h_{j+1} + h_{j-1}^2$;

(iv) if $\gamma(z_j) = s_{j-1}$ and $\gamma(z_{j+1}) = s_j$, then $\zeta_j = h_{j-1}d_{j+1} = h_{j-1}h_j + h_{j-1}^2$;

We call g_j *slanting* if it is as in case (iii) or (iv) of (6.9). If no edge of C is slanting, then γ is either γ_1 or γ_2. Otherwise C contains at least two slanting edges. In this case we observe from (6.9) that the representation of ζ^γ includes $h_i^2 + h_j^2$ (or $2h_i^2$) for some i, j, which is at least α. Now the result follows from the fact that the representation includes $2h_ih_j$ for each $0 \leq i < j \leq 2$.

To see the latter, w.l.o.g., assume $i = 0$, $j = 2$, and consider the edges g_0 and g_1. By (6.6), g_0 contributes h_0h_2 in cases (ii),(iv), i.e., when $\gamma(z_1) = s_0$. And if $\gamma(z_1) = s_2$, then g_1 contributes h_0h_2. Similarly, the pair g_3, g_4 contributes h_0h_2. ∎

This completes the proof of Theorem 1.6.

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