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Abstract. There are various versions of the multicommodity flow problems such as edge-demand, node-demand, maximization and locking problems. Here we reveal some relationships between these problems. In particular, with an unexpected use of Edmonds' polymatroid intersection theorem, we derive a theorem of Karzanov and Lomonosov on multiflow maximization. The approach gives rise to a (combinatorial) polynomial time algorithm to find the maximum in question. A certain weighted version of the maximization problem also becomes tractable.

1. INTRODUCTION

Let $G = (V, E)$ and $H = (T, F)$ be two graphs so that $T \subseteq V$. We call a path of G *H-admissible* if it connects two nodes x, y of T so that $xy \in F$. G will be called a *supply graph*, H a *demand graph* and the elements of T *terminals* while the other elements of V are called *inner nodes.*

There are various forms of the edge-disjoint paths problem. In the *edge-demand problem* we are given a demand function $m : F \to \mathbf{Z}_+$ and the objective is to find a family $\mathcal{F}$ of edge-disjoint H-admissible paths so that for each demand edge $f = xy$ there are $m(f)$ members of $\mathcal{F}$ connecting x and y. The *maximization problem* consists of finding a maximum number of edge-disjoint H-admissible paths.

If H consists of one edge, then the Menger theorem answers both questions. In the example shown in Figure 1 H consists of two disjoint edges. Clearly, there is no solution to the

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edge-demand problem when both demands are 1. The maximum value in the maximization problem is 2 since there are two paths connecting the end-nodes of one of the two demand edges.

In the edge-demand problem the *cut condition* is a simple necessary condition: for every cut, the number of edges in G cannot be smaller than the sum of demands in this cut.

In general, both problems are NP-complete even in the special case when G is Eulerian. However there are important special cases when the problems are tractable. For example, Rothschild and Whinston, sharpening earlier results of Hu, proved that in the edge-demand problem the cut condition is sufficient when H consists of two edges, and for each node v of G $d(v) + \sum(m(uv) : uv \in E)$ is even where $d(v)$ denotes the number of edges of G incident to v.

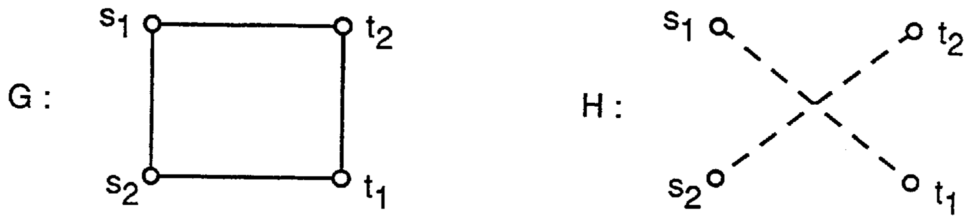

Fig. 1

The same theorem holds if H consists of two stars (that follows by an easy elementary construction from the theorem of Rothschild and Whinston [1966b]), if H is K_4 (complete graph on four nodes) [Lomonosov, 1979; Seymour, 1980] and if H is C_5 (5-element circuit) [Lomonosov, 1979]. It is also known that there is no any other demand graph for which the statment above holds. For a detailed account on edge-disjoint paths problems see [Frank, 1990].

As to the maximization problem, Rothschild and Whinston [1966a] proved a max-flow min-cut type theorem when G is *inner Eulerian* (that is, $d(v)$ is even for every inner node v) and H consists of two edges. Another important theorem is due, independently, to L. Lovász [1976] and B.V. Cherkasskij [1977] . They solve the maximization problem when H is a complete graph and G is inner Eulerian. In [1978] A.V. Karzanov and M.V. Lomonosov found (in a sense, a strongest) common generalization of these two theorems. Their original proof is rather lengthy and technical and it is certainly much more difficult than those of the two special cases mentioned above.

The main contribution of this paper is a relatively simple proof of the theorem of Karzanov and Lomonosov. The proof relies on two ingredients. First we provide a simple proof of the so-called locking theorem, another earlier result of Karzanov and Lomonosov. In the second step we invoke the locking theorem and the polymatroid intersection theorem of J.Edmonds [1970] . Both of these ingredients can be efficiently solved; this gives rise to a combinatorial strongly polynomial time algorithm for the (capacitated) maximization problem in question, which has significally less running time than an algorithm in [Karzanov, 1985] (described in details in [Karzanov, 1987]) based on splitting-off techniques.

In what follows we do not distinguish between a one-element set $\{x\}$ and its only element x. For a set X and an element t let $X + t$ denote the union of X and t. For a vector $m : S \to \mathbf{R}$ and $X \subseteq S$ we use the notation $m(X)$ for $\sum(m(s) : s \in X)$. A family of pairwise disjoint non-empty subsets of a set S is called a *sub-partition* of S. For two elements s, t a set X is called a *$t\bar{s}$-set* if $t \in X, s \notin X$. An integer-valued vector or function is called *even* if each of its values is an even integer. For a polyhedron P we use the notation $P/2 := \{x : 2x \in P\}$.

For a graph $G = (V, E)$ the cut $[X, V - X]$ is the set of edges with precisely one end-node in X. Its cardinality is denoted by $d(X)(= d(V - X))$. $d(X)$ is called the *degree function* of G. Let $d(X, Y)$ denote the number of edges with one end in $X - Y$ and the other in $Y - X$. It is easy to prove that d satisfies the following identities for every pair X, Y of subsets of V.

$$d(X) + d(Y) = d(X \cap Y) + d(X \cup Y) + 2d(X, Y) \tag{1.1}$$

$$d(X) + d(Y) = d(X - Y) + d(Y - X) + 2d(X \cap Y, V - (X \cup Y)) \tag{1.2}$$

Splitting off a pair of adjacent edges $e = st, f = sx$ means an operation that replaces e and f by a new edge connecting x and t (this way we may introduce parallel edges between x and t.) The resulting graph is denoted by G^{ef}.

Let A and B be two disjoint subsets of V. A path connecting an element of A and an element of B is called an *(A, B)-path.* A path connecting two distinct elements of A is called an *A-path.* $\lambda(A, B; G)$ or simply $\lambda(A, B)$ stands for the maximum number of edge-disjoint (A, B)-paths. By Menger's theorem $\lambda(A, B) = \min(d(X) : A \subseteq X \subseteq B)$.

One may consider a fractional version of the edge-disjoint paths problem. Let G and H be as before. By an *H-multiflow* or briefly *multiflow* x we mean a family $\{P_1, P_2, \ldots, P_k\}$ of paths of G along with non-negative coefficients $\alpha_1, \alpha_2, \ldots, \alpha_k$ so that each P_i connects the end-nodes of a demand edge. x is *integer-valued* if each α_i is an integer.

If each P_i connects an element of A and element of B (in particular, when H is a complete bipartite graph with bipartition (A, B)), we speak of an (A, B)-*flow*. For an H-multiflow x let $x(e) := \sum(\alpha_i : P_i \text{ uses } e)$ $(e \in E)$ and $x(t) := \sum(\alpha_i : P_i \text{ ends at } t)$ $(t \in T)$. For a given *capacity function* $c : E \to \mathbf{R}_+$, x is called c-*admissible* if $x(e) \leq c(e)$ for every $e \in E$.

A non-negative set-function $b : 2^T \to \mathbf{R}_+$ is called a *polymatroid function* if

1. $b(\emptyset) = 0$,
2. b is monotone increasing, i. e. $b(X) \geq b(Y)$ when $Y \subseteq X \subseteq T$,
3. b is submodular, i.e. $b(X) + b(Y) \geq b(X \cup Y) + b(X \cap Y)$ for $X, Y \subseteq T$.

The degree-function d of a graph G satisfies properties 1 and 3 but typically not 2.

A polyhedron $P(b) := \{x \in R^T : x \geq 0, \ x(A) \leq b(A) \text{ for every } A \subseteq T\}$ is called a *polymatroid.* It is called *integral* if every vertex of P is integer-valued.

The concept of a polymatroid was introduced by J. Edmonds. He proved several fundamental theorems concerning polymatroids. For example, he proved that a polymatroid uniquely determines its defining polymatroid function (in other words, different polymatroid functions define different polymatroids.) Furthermore, a polymatroid is integral if and only if b is integer-valued.

For a polymatroid $P(b)$ the face $B(b) := \{x : x \in P, \ x(T) = b(T)\}$ of $P(b)$ is called the *basis polyhedron* of $P(b)$ and its elements are the *bases.* Edmonds also proved the following important feature of polymatroids.

Theorem 1.1 [Edmonds, 1970]. *For an (integral) polymatroid $P(b)$ and an (integer-valued) vector $x \in P(b)$ there is an (integer-valued) basis with $y \geq x$.*

Probably the most important result of Edmonds concerning polymatroids is the Intersection Theorem.

Theorem 1.2 [Edmonds, 1970]. *For two polymatroid functions a and b defined on the power set of T*

$$\max(x(T) : x \in P(a) \cap P(b)) = \min(a(X) + b(T - X) : X \subseteq T).$$

Furthermore, if a and b are integer-valued, the maximum is attained by an integer vector.

It follows that there is a vector x in $P(a) \cap P(b)$ and a bipartition $\{A, B\}$ of T so that $x(A) = a(A)$ and $x(B) = b(B)$ and if a and b are integral-valued, then so is x.

2. THE LOCKING PROBLEM

For a subset $A \subseteq T$ the notation $\lambda(A, T-A; G)$ will be abbreviated by $\lambda(A; G)$ or by $\lambda(A)$ when no confusion can arise. Throughout this section we assume that the pair (G, T) is inner Eulerian.

Lovász [1976] and Cherkasskij [1977] proved the following theorem.

Theorem 2.1. *For an inner Eulerian pair (G, T) the maximum number of edge-disjoint T-paths is equal to $\sum(\lambda(t) : t \in T)/2$. Furthermore, there is a family of disjoint sets $\{X(t) : t \in T\}$ so that $t \in X(t) \subseteq V$ and $d(X(t)) = \lambda(t)$ for $t \in T$.*

An equivalent formulation of the first part is:

Theorem 2.1'. *There is a family $\mathcal{F}$ of edge-disjoint T-paths so that $\mathcal{F}$ contains $\lambda(t)$ paths ending at t for each $t \in T$.*

In other words, there is a family of edge-disjoint T paths that contains maximal families of edge-disjoint $(t, T-t)$-paths simultaneously for all $t \in T$.

Karzanov and Lomonosov extended this theorem. To formulate their result let us say that a family $\mathcal{F}$ of edge-disjoint T-paths *locks a subset* $A \subseteq T$ if $\mathcal{F}$ contains $\lambda(A)$ paths connecting A and $T - A$. Furthemore, we say that $\mathcal{F}$ *locks a family* $\mathcal{L}$ of subsets of T if $\mathcal{F}$ locks all members of $\mathcal{L}$.

Theorem 2.1' asserts that there is a family $\mathcal{F}$ of paths that locks all singletons of T. Is it always possible to find a family of edge-disjoint T-paths that locks a specified family $\mathcal{L}$? The answer is, in general, negative, as shown by the following two instances. Here in each instance $\mathcal{L}$ consists of three pairwise crossing sets. (Two subsets A and B of T are called *crossing* if none of $A - B, B - A, A \cap B, T - (A \cup B)$ is empty).

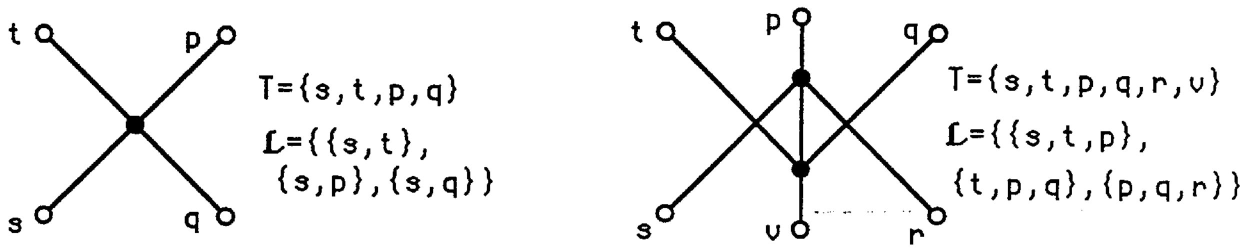

Fig. 2

The theorem below asserts essentially that if we exclude these configurations, then a locking T-path system always exists.

Call a family $\mathcal{L}$ of subsets of T *3-cross free* if it has no three pairwise crossing members.

Locking Theorem 2.2 [Karzanov, 1984; Lomonosov, 1985]. *Let (G,T) be inner Eulerian and $\mathcal{L}$ a 3-cross free family of subsets of T. Then there exists a family of edge-disjoint T-paths that locks $\mathcal{L}$.*

A proof of a slightly weaker version was sketched in [Karzanov and Lomonosov, 1978]. The present proof relies on an idea used already in [Karzanov, 1984] but technically simpler.

Proof. We may assume that $T - A \in \mathcal{L}$ for each $A \in \mathcal{L}$ because for $A \in \mathcal{L}$ adding $T - A$ to $\mathcal{L}$ affects neither 3-cross-freeness nor lockability. Also assume that G is connected.

We proceed by induction on the number of edges incident to the elements of $V - T$. If this number is zero, then the statement is trivial. Therefore there is an edge $e = st$ with $t \in T, s \notin T$. We are going to show that there is an edge $f = sx$, $x \neq t$, for which

$$\lambda(A;G) = \lambda(A;G^{ef}) \text{ for every } A \in \mathcal{L}. \tag{2.1}$$

From this the theorem follows since, by induction, there is a family $\mathcal{F}$ of T-paths of G^{ef} locking $\mathcal{L}$. If a path $P \in \mathcal{F}$ uses the new edge h of G^{ef} having arisen from the splitting of e, f, then revise $\mathcal{F}$ by replacing h in P by e and f. By (2.1) the revised $\mathcal{F}$ locks $\mathcal{L}$ in G.

Claim 1. *Suppose for $X, Y \subseteq V$ that $X \cap T \subseteq Y \cap T$ and that $d(X) = \lambda(X \cap T), d(Y) = \lambda(Y \cap T)$. Then $d(X \cap Y) = \lambda(X \cap T), d(X \cup Y) = \lambda(Y \cap T)$ and $d(X,Y) = 0$.*

Proof. Since $X \cap T \subseteq Y \cap T$ we have $(X \cap Y) \cap T = X \cap T$ and hence $d(X \cap Y) \geq \lambda(X \cap T)$. Analogously, $(X \cup Y) \cap T = Y \cap T$ and $d(X \cup Y) \geq \lambda(Y \cap T)$. Therefore, by (1.1), $\lambda(X \cap T) + \lambda(Y \cap T) = d(X) + d(Y) = d(X \cap Y) + d(X \cup Y) + 2d(X, Y) \geq \lambda(X \cap T) + \lambda(Y \cap T) + 2d(X, Y)$, from which the claim follows. •

Call a set $X \subseteq V$ *tight* if $X \cap T \in \mathcal{L}$ and $d(X \cap T) = \lambda(X \cap T)$. Since $\mathcal{L}$ is closed under complementation, $V - X$ is tight if X is tight. Because (G, T) is inner Eulerian, a pair of edges $e = st, f = sx$ will satisfy (2.1) precisely if

$$\text{there is no tight set } X \text{ with } t, x \in X \subseteq V - s. \tag{2.2}$$

Claim 2. *There are no three maximal tight $t\bar{s}$-sets.*

Proof. Let X, Y, Z be maximal tight $t\bar{s}$-sets. Since $\mathcal{L}$ is 3-cross-free, two of the tree sets $X \cap T, Y \cap T, Z \cap T$, say $X \cap T$ and $Y \cap T$, are non-crossing.

Then either $X \cap T \subseteq Y \cap T$ or $Y \cap T \subseteq X \cap T$ or $T \subseteq X \cup Y$. In the first two cases Claim 1 implies that $X \cup Y$ is tight contradicting the maximality of X and Y. In the last case, by applying Claim 1 to $X' = V - X$ and Y, we obtain that $d(X', Y) = 0$ contradicting the existence of edge st. •

Let S denote the set of neighbours of s.

Claim 3. *It is not possible to cover S by two tight $t\bar{s}$-sets.*

Proof. Suppose that $S \subseteq X \cup Y$ where X and Y are tight $t\bar{s}$-sets. Let $\alpha := d(s, X - Y), \beta := d(s, Y - X), \gamma := d(s, X \cap Y)$. By symmetry we may assume that $\alpha \geq \beta$. $(X + s) \cap T = X \cap T$ implies that $d(X + s) \geq \lambda(X \cap T)$. On the other hand, since γ is positive, we have $d(X + s) = d(X) - \alpha - \gamma + \beta < d(X) = \lambda(X \cap T)$, a contradiction. •

By Claims 2 and 3 there is an edge $f = sx$ satisfying (2.2) and then (2.1) holds; the proof of Locking Theorem is complete. • • •

We will need a slight extension of Theorem 2.2. Let $m : T \to \mathbf{Z}$ be a non-negative integer-valued function on T. A family $\mathcal{F}$ of edge-disjoint T-paths is called *m-independent if* every terminal $t \in T$ is the end of at most $m(t)$ members of $\mathcal{F}$. Let $\lambda_m(A)$ denote the maximum number of m-independent $(A, T-A)$-paths. We say that a family $\mathcal{F}$ of T-paths *m-locks a subset* $A \subseteq T$ if $\mathcal{F}$ is m-independent and contains $\lambda_m(A)$ $(A, T-A)$-paths. Furthemore, we say that $\mathcal{F}$ *m-locks a family* $\mathcal{L}$ of subsets of T if $\mathcal{F}$ m-locks all members of $\mathcal{L}$.

The following theorem is a straightforward consequence of Theorem 2.2 and will be used in the proof of Theorem 4.3.

Theorem 2.3. *Let G be Eulerian and $\mathcal{L}$ a 3-cross free family of subsets of T. Let $m : T \to \mathbf{Z}_+$ be an even vector. Then there exists a family $\mathcal{F}$ of T-paths that m-locks $\mathcal{L}$.*

Proof. Let T' be a copy of T so that $T' \cap V = \emptyset$ and let t' denote the the element of T' corresponding to $t \in T$. Let $G' = (V', E')$ where $V' := V \cup T'$ and $E' := E \cup \{m(t)$ parallel edges between t and t' for every $t \in T\}$. Let $\mathcal{L}' := \{\{t' : t \in L\} : L \in \mathcal{L}\}$. Apply Theorem 2.2 to G', T' and $\mathcal{L}'$. • • •

3. FLOWS AND POLYMATROIDS

Let $c : E \to \mathbf{R}_+$ be a capacity function on the edges of $G = (V, E)$. Let $A \subset T$ and $B := T - A$. Define $P_A := \{m \in \mathbf{R}_+^A$: there is a c-admissible (A, B)-flow x such that $x(v) = m(v)$ for $v \in A\}$.

For $X \subseteq A$ let $f_A(X) := \min(\delta_c(Y) : Y \subseteq V, X \subseteq Y \cap T \subseteq A)$. Here $\delta_c(Y) := \sum(c(e) : e \in [Y, V - Y]\ )$. Clearly f_A is submodular and monotone increasing. By a version of the Max-flow Min-cut theorem a vector $m \in R_+^A$ belongs to P_A if and only if $m(X) \leq f_A(X)$. Therefore P_A is a polymatroid. Furthermore, if c and m are integer-valued, then there is a c-admissible integer-valued (A, B)-flow x for which $x(v) = m(v)$.

Let G be Eulerian. Define c by $c(e) = 1$ for every $e \in E$ and let $\mathcal{T} := \{T_1, T_k, \ldots, T_k\}$ be a partition of T.

Let P denote the direct sum of polymatroids $P_{T_1}, P_{T_2}, \ldots, P_{T_k}$. Since G is Eulerian, $P/2$ is an integral polymatroid.

Lemma 3.1. *Let q be an even basis of P and m an even vector for which $m \geq q$. Then any m-independent family $\mathcal{F}$ of T-paths that m-locks $\mathcal{T}$ contains at least $q(T)/2$ paths connecting distinct members of $\mathcal{T}$.*

Proof. Since q is a basis, there are $q(T_i)$ edge-disjoint paths connecting T_i and $T - T_i$ so that each $t \in T$ is the end-node of precisely $q(t)$ paths. Therefore $\lambda_q(T_i) = q(T_i)$. The assumption $m \geq q$ implies that $\lambda_m \geq \lambda_q$. Since $\mathcal{F}$ m-locks $\mathcal{T}$, there are $\lambda_m(T_i) \geq \lambda_q(T_i) = q(T_i)$ paths in $\mathcal{F}$ connecting T_i and $T - T_i$ for each $i = 1, \ldots, k$, from which the lemma follows. •

Remark. Since q is a basis, $\mathcal{F}$ contains at most $q(T_i)$ $(T_i, T - T_i)$-paths and therefore $\mathcal{F}$ contains at most $q(T)/2$ paths connecting distinct members of $\mathcal{T}$. That is, the number of such paths in $\mathcal{F}$ is precisely $q(T)/2$ but we will not need this fact.

4. MAXIMIZATION

Let $G = (V, E)$ and $H = (T, F)$ be two graphs so that $T \subseteq V$ and $E \cap F = \emptyset$. Throughout this section we assume that the pair (G, T) is inner Eulerian, that is, $d(v)$ is even for every $v \in V - T$ where d stands for the degree function of G.

The *maximization form* of the edge-disjoint paths problem consists in finding a maximum number $\mu = \mu(G, H)$ of edge-disjoint H-admissible paths in G.

We can easily get an upper bound on μ. Let us call a sub-partition $\{X_1, X_2, \ldots, X_k\}$ of V *admissible* if $T \subseteq \cup X_i$ and each $X_i \cap T$ is stable in $H (i = 1, \ldots, k)$. Clearly,

$$\mu(G, H) \leq \sum d(X_i)/2. \tag{4.1}$$

Let us call $\sum d(X_i)/2$ the *value* of the sub-partition. Let $\tau = \tau(G, H)$ denote the minimum value of an admissible sub-partition. Then $\mu \leq \tau$.

The following example shows that we do not have equality, in general.

Fig. 3

There are two known special cases when equality holds. Theorem 2.1 shows that this is the case if H is a complete graph on T. Reformulating Theorem 2.1 we have:

Theorem 4.1. *Suppose that (G,T) is inner Eulerian and the demand graph H is complete. Then $\mu(G,H) = \tau(G,H)$.* •

Another special case for which $\mu = \tau$ is when H consists of two edges, that is, $H = 2K_2$.

Theorem 4.2. *Suppose that (G,T) is inner Eulerian and H consists of two edges s_it_i ($i = 1,2$). Then $\mu(G,H) = \tau(G,H)$.* •

This is a theorem of Rothschild and Whinston. Actually, they proved it in the following simpler form (it is easy to prove that $\tau' = \tau$).

Theorem 4.2' [Rothschild and Whinston, 1966a]. *Suppose that (G,T) is inner Eulerian and H consists of two edges s_it_i ($i = 1,2$). Then $\mu(G,H)$ is the minimum cardinality τ' of a cut $[X, V - X]$ of G for which $\{s_i,t_i\} \cap X = 1$ ($i = 1,2$).* •

Let us call a graph $H = (T,F)$ *bi-stable* if the family of maximal stable sets of H can be partitioned into two parts, each consisting of disjoint sets. (It can be shown that the bi-stable graphs are precisely the complements of the line graph of bipartite graphs. Moreover, it is easy to design a polynomial time algorithm to recognize if a graph is bi-stable and, if so, to construct a partition as above.) Clearly, a clique, or more generally a complete k-partite graph is bi-stable and $2K_2$ is bi-stable as well. Therefore Theorems 2.1 and 2.2 are special cases of the following.

Theorem 4.3 [Karzanov, 1985; Lomonosov, 1985]. *Suppose that (G,T) is inner Eulerian and $H=(T,F)$ is bi-stable. Then $\mu(G,H)=\tau(G,H)$.*

A proof of a slightly weaker version was sketched in [Karzanov and Lomonosov, 1978]. The reader may feel that bi-stable demand graphs form a rather peculiar class of graphs and there may be a larger, more natural class of graphs for which $\mu=\tau$ holds. Karzanov and Pevzner [1979], however, showed that if $H=(T,F)$ contains no isolated nodes and is not bi-stable, then there is a supply graph $G=(V,E)$ with $T\subseteq V$ such that even for the numbers $\mu^*(G,H)$ and $\nu^*(G,H)$ in the corresponding fractional relaxations one has $\mu^*(G,H)<\nu^*(G,H)$. We now give a new, short, proof of Theorem 4.3 using results obtained in Sections 2 and 3.

Proof. By (4.1) we have $\mu(G,H)\leq\tau(G,H)$. To see the inverse inequality, first we prove that the theorem follows from its special case when the graph is totally Eulerian. So suppose the theorem is true for (G',H') whenever G' is Eulerian and we want to prove it for (G,H) when G is inner Eulerian. Let K denote the set of nodes of G with odd degree. Since (G,T) is inner Eulerian $K\subseteq T$. If K is empty, we are done. If not, for a new node t, let $T':=T+t$ and $V':=V+t$. Let $E':=E\cup\{xt:x\in K\}$ and $F':=F\cup\{xt:x\in T\}$. Then $G':=(V',E')$ is Eulerian and $H':=(T',F')$ is bi-stable. Let μ' and τ' denote, respectively, the maximum and minimum in question concerning (G',H'). By the assumption $\mu'=\tau'$.

Obviously, there is an optimal solution to the maximization problem concerning (G',H') in which every edge xt, $x\in K$, is itself a path in the solution. Thus we have $\mu\geq\mu'-|K|$. Furthermore, let $\mathcal{M}'$ be an optimal admissible sub-partition for (G',H') so that $t\in X\in\mathcal{M}'$. Since every edge xt, $x\in T$, belongs to H', $X\cap T=\{t\}$. Hence $\mathcal{M}'-\{X\}$ is an admissible sub-partition for (G,H); therefore $\tau\leq\tau'-|K|$. We conclude that $\mu\geq\mu'-|K|=\tau'-|K|\geq\tau$, as required.

Henceforth we assume that G is Eulerian. Let $\mathcal{A}'=\{A'_1,\ldots,A'_h\}$ and $\mathcal{B}'=\{B'_1,\ldots,B'_k\}$ be two families of maximal stable sets of H that give a partition as above. Let $\mathcal{A}:=\mathcal{A}'\cup\{\{t\}:t\in T-\cup A'_i\}$ and $\mathcal{B}:=\mathcal{B}'\cup\{\{t\}:t\in T-\cup B'_i\}$. Then each of $\mathcal{A},\mathcal{B}$ forms a partition of T and consists of disjoint (not necessarily maximal) stable sets.

For $A_i\in\mathcal{A}$ let $a_i(X)$ $(X\subseteq A_i)$ be the set-function defined by $a_i(X):=\lambda(X,T-A_i)$. We have seen in Section 3 that a_i is a polymatroid function. Define b_j analogously for $\mathcal{B}$. For $X\subseteq T$ let $a(X):=\sum a_i(X\cap A_i)$ and $b(X):=\sum b_j(X\cap B_j)$. Then a and b are polymatroid functions, as mentioned in Section 3. Let $P(a)$ and $P(b)$ be the polymatroids defined by a and b.

Since G is Eulerian, $a/2$ and $b/2$ are integer vectors and therefore $P(a)/2$ $(= P(a/2))$ and $P(b)/2$ are integral polymatroids. By Theorem 1.2 there exist an even vector m' in $P(a) \cap P(b)$ and a bi-partition $\{A, B\}$ of T so that

$$m'(A) = a(A) \text{ and } m'(B) = b(B). \tag{4.2}$$

By Theorem 1.1 there are even bases $m_a \in P(a)$ and $m_b \in P(b)$ so that $m_a \geq m'$ and $m_b \geq m'$. By (4. 2) we have

$$m'(t) = m_a(t) \text{ if } t \in A \text{ and } m'(t) = m_b(t) \text{ if } t \in B. \tag{4.3}$$

Let a vector m be defined by $m(t) := \max(m_a(t), m_b(t))$ for $t \in T$. Clearly m is even and (4.3) implies that $m_a(t) + m_b(t) = m(t) + m'(t)$ for each $t \in T$, hence

$$m_a(T) + m_b(T) - m(T) = m'(T). \tag{4.4}$$

Let $\mathcal{L} := \mathcal{A} \cup \mathcal{B}$. Clearly $\mathcal{L}$ is 3-cross free so we can apply Theorem 2.3. Let $\mathcal{F}$ be a family of m-independent T-paths provided by the theorem. Recall that a path was called H-admissible if it connects the end-nodes of a demand edge.

Claim 1. *The number h of H-admissible paths in $\mathcal{F}$ is at least $m'(T)/2$.*

Proof. Note that a path is not H-admissible precisely if it connects two nodes belonging to the same member of $\mathcal{L}$ $(= \mathcal{A} \cup \mathcal{B})$. Apply Lemma 3.1 with the choice $\mathcal{T} := \mathcal{A}$, $P := P(a)$ and $q := m_a$.

We obtain that there are at most $|\mathcal{F}| - m_a(T)/2$ paths in $\mathcal{F}$ having both end-nodes in the same member of $\mathcal{A}$. Analogously, there are at most $|\mathcal{F}| - m_b(T)/2$ paths in $\mathcal{F}$ having both end-nodes in the same member of $\mathcal{B}$.

Hence $h \geq |\mathcal{F}| - (|\mathcal{F}| - m_a(T)/2) - (|\mathcal{F}| - m_b(T)/2) = (m_a(T) + m_b(T))/2 - |\mathcal{F}| \geq (m_a(T) + m_b(T) - m(T))/2 = m'(T)/2$, as required. Here the inequality follows since $\mathcal{F}$ is m-independent while the last equality is precisely (4.4). •

Claim 2. *There is an admissible sub-partition of value at most* $(a(A)+b(B))/2$.

We leave the proof to the reader.

By Claims 1 and 2 and by (4.2) we have $\mu \geq m'(T)/2 = (a(A)+b(B))/2 \geq \tau$, and the proof of Theorem 4.3 is complete. • • •

5 ALGORITHMIC ASPECTS

In this section we briefly outline how the proof method above gives rise to a strongly polynomial (combinatorial) algorithm in the capacitated case. (Informally, a polynomial-time algorihm is *strongly polynomial* if the number of steps does not depend on the magnitude of the occuring capacities).

The input of the algorithm consists of two graphs $G = (V, E)$ and $H = (T, F)$ where $T \subseteq V$. G is endowed with a non-negative rational capacity function $c : E \to \mathbf{Q}_+$. We assume that $H = (T, F)$ is given by two partitions $\mathcal{A}= \{A_1, A_2, \dots, A_h\}$ and $\mathcal{B}= \{B_1, B_2, \dots, B_k\}$ of T so that $xy \in F$ if and only if x and y do not belong to the same A_i and to the same B_j.

The output of the algorithm consists of an admissible sub-partition $\mathcal{K}= \{X_1 X_2, \dots, X_t\}$ of V and a c-admissible H-multiflow x so that $\sum(x(t) : t \in T) = \sum(\delta_c(X) : X \in \mathcal{K})$. Moreover, if c is is integer-valued and *Eulerian* in the sense that $\delta_c(v)$ is even for every node $v \in V$, then the output x is integer-valued.

Actually, we will assume that c is integer-valued and Eulerian. If this is not the case, one can multiply through the capacities by $2N$ where N denotes the common denominator of the capacities.

First we remark that the proof of the Locking Theorem immediately provides a polynomial-time algorithm for the set-system $\mathcal{A} \cup \mathcal{B}$ when c is identically 1. It is not difficult to show that, for general c, if in every step one splits off as much as possible, then the algorithm is strongly polynomial. In what follows we comment on the use of the polymatroid intersection algorithm to construct an even vector m' and admissible sub-partition occuring in the proof of Theorem 4.3.

For disjoint sets $X, Y \subseteq V$ let $\lambda_c(X, Y)$ denote the maximum value of a flow between X and Y. With the help of a Max-flow Min-cut (MFMC) computation $\lambda_c(X, Y)$ can be computed in (strongly) polynomial time.

For $A_i \in \mathcal{A}$ let $a_i(X)$ $(X \subseteq A_i)$ be a set function defined by $a_i(X) := \lambda_c(X, T - A_i)$. Define b_j analogously. For $X \subseteq T$ let $a(X) := \sum a_i(X \cap A_i)$ and $b(X) := \sum b_j(X \cap B_j)$. Let $P(a)$ and $P(b)$ be the polymatroids defined by a and b. Since c is Eulerian, $P(a)/2$ and $P(b)/2$ are integral polymatroids.

The Polymatroid Intersection Theorem 1.2 ensures the existence of an even vector m' in $P(a) \cap P(b)$ and a bi-partition $\{A, B\}$ of T so that $m'(A) = a(A)$ and $m'(B) = b(B)$. P. Schönsleben [1980] developed a strongly polynomial algorithm for determining m' and $\{A, B\}$. His algorithm works if an oracle is available to minimize $a(X) - z(X)$ and $b(X) - z(X)$ over $X \subseteq T$ where $z : T \to \mathbf{Q}$ is a vector. In our case this oracle can indeed be constructed by invoking the MFMC algorithm.

There is, however, a better way to compute m' and $\{A, B\}$ by exploiting the special structure of the two polymatroids above. We now describe such a method that relies directly on the MFMC computation and avoids the use of a general–purpose polymatroid intersection algorithm.

Define a mixed graph $G_{big} = (V_{big}, E_{big})$ as follows (this graph is a slight modification of one designed in [Karzanov, 1985], however, the former enables us to determine easily m' and (A, B), while the latter only a partition (A, B)). First construct $h + k$ disjoint copies of G and denote them by $G_{A_1}, \dots, G_{A_h}, G_{B_1}, \dots, G_{B_k}$. Add then two new nodes s, t.

For every i, j $(1 \le i \le h, 1 \le j \le k)$ and every $v \in T$, define edges, as follows. If $v \in T - A_i$, connect s by an edge with the copy of v in G_{A_i}. If $v \in T - B_j$, connect t by an edge with the copy of v in G_{B_j}. Finally, if $v \in A_i \cap B_j$, draw a *directed* edge from the copy of v in G_{A_i} to the copy of v in G_{B_j}; the edge defined this way is denoted by $e(v)$.

Define a capacity fuction $c : E_{big} \to \mathbf{Z}_+ \cup \{\infty\}$, as follows. If an edge e' of G_{big} is a copy of an edge e of G, then let $c(e') := c(e)$. The capacity of all other edges of G_{big} is ∞. Let M denote the value of a maximum flow from s to t.

Let $R \subset V_{big}$ be an $s\bar{t}$-set for which $\delta_c(X) = M$. Define X_i (respectively, Y_j) to be the set of nodes in G for which the corresponding nodes in G_{A_i} (G_{B_j}) do not belong to R (respectively, belong to R). Finally, using that c is Eulerianone can determine a maximum flow from s to t in such a way that for every $v \in T$ the value of this flow on $e(v)$ is even; denote this value by $m'(v)$.

It is easy to prove that $m' \in P(a) \cap P(b)$ and $m'(T) = M$. Moreover, applying to $\mathcal{K} := \{X_1, \dots, X_h, Y_1, \dots, Y_k\}$ "uncrossing" operations we obtain an optimal admissible sub-partition of value $m'(T)/2$.

The running time of the above method to construct m' is $0(\eta(|V_{big}|)$, or $O(\eta(|V||T|)$ operations, where $\eta(n)$ is complexity of finding a maximum in a graph with n vertices. One can show that to determine an even basis m_a as in the above proof of Theorem 4.3 (for the capacitated version) is reduced to finding $|\mathcal{A}|$ maximum flows in G; and similarly for m_b. Thus m as in (4.4) is determined in running time $O(|T|\eta(|V|))$. Finally, one shows that the locking problem (for m) can be solved by use of only $O(|V|)$ splitting-off operations for every vertex in $V - T$, each operation consists in finding a maximum flow in G. This gives running time $O(|V|^2\eta(|V|))$ to solve the locking problem, and $O(|V|^3(|V|^2 + |T|^3))$ for the whole algorithm if for determining a maximum flow we use a subroutine of complexity $\eta(n) = O(n^3)$. Note that the algorithm in [Karzanov, 1985] requires running time $O(|T|^3|V|^6)$.

6. NODE-DEMAND PROBLEM

The reader might have a feeling that there is a seemingly unnecessary twist in the proof above. Recall that the polymatroid intersection theorem ensured the existence of a maximum even vector m' in $P(a) \cap P(b)$ for which $m'(T)/2$ is precisely $\tau(G,H)$. Is it indeed necessary to make a detour to the locking theorem or it would perhaps be possible to use m' directly to construct $m'(T)/2$ H-admissible paths?

This would be the case if there existed a system of H-admissible paths so that each $t \in T$ is the end-node of precisely $m'(t)$ of them. Unfortunately, such a system need not always exist but the following problem naturally emerges.

Let $G = (V,E)$ be a graph $H = (T,F)$ a demand graph with $T \subseteq V$. Moreover, let $m : T \to \mathbf{Z}_+$ be a *demand* function.

The *node-demand problem* consists in finding a system of H-admissible paths so that each terminal t is the end-node of precisely $m(t)$ paths. We call the problem and also the vector m *feasible* when such a solution exists.

The node-demand problem is called *Eulerian* if (G,T) is inner Eulerian and $m(t) + d(t)$ is even for each $t \in T$. We call a demand graph H *two-covered* (*one-covered*) if every node $t \in T$ belongs to at most two (exactly one) maximal stable sets of H; in particular, if H is bi-stable, then it is two-covered. (The converse is, in general, not true).

Theorem 6.1. *Suppose that the node-demand problem defined by (G,H,m) is Eulerian* *and H is two-covered. Then it is feasible if and only if the following node-cut condition*

$$m(X \cap S) - m(X \cap T - S) \leq d(X) \tag{6.1}$$

holds for every $X \subseteq V$ and every maximal stable set S in H.

Sketch of Proof. The necessity of the node-demand problem is straightforward. The proof of sufficiency relies on the observation that the family $\mathcal{L}$ of maximal stable sets of a 2-covered graph H is 3-cross-free. Then we apply Theorem 2.3 and prove that the path system provided by this theorem gives the required solution to the node-demand problem, by showing that (6.1) implies that $\lambda_m(X) = m(X)$ for every $X \in \mathcal{L}$. •

Several authors introduced a generalization of polymatroids called by A. Bouchet and W. Cunningham [1991] bi-submodular polyhedra. This is a general class of polyhedra for which a naturally formulated general greedy algorithm works. A bi-submodular polyhedron can be defined as a polyhedron of the form

$$\{x \in \mathbf{R}^T : x(M) - x(N) \leq b(M,N) \quad \text{for } M, N \subseteq T,\ M \cap N = \emptyset\}, \tag{6.2}$$

where b is a real-valued function defined on all pairs (M, N) of disjoint sets $M, N \subseteq T$ satisfying the "bi-submodular inequalities":

$$b(M,N) + b(M',N') \geq b(M \cap M', N \cap N') + b((M \cup M') - (N \cup N'), (N \cup N') - (M \cup M')).$$

THEOREM 6.2 *Suppose that H is one-covered. Then the polyhedron*
$\{x : x(X \cap S) - x(X \cap T - S) \leq d(X)$ for every $X \subseteq V$ and S maximal stable set in H $\}$ is a bi-submodular polyhedron.

Theorem 6.2 can be proved with a routine "truncation procedure" which consists in defining a function b on every pair of disjoint subsets of T in such a way that $\{x : x(A) - x(B) \leq b(A, B)$ for every $A, B \subseteq V,\ A \cap B = \emptyset\ \}$ is the same polyhedron as the one in the theorem, and $b(A, B)$ is as small as possible. The bi-submodularity of b can be proved using (1.1) and (1.2). Moreover, b is integral, and it is even whenever d is even.

Let us consider again bi-stable demand graphs. Then by Theorem 6.1 the set of feasible node-demands m is exactly the set of solutions of (6.1). But the inequalities in (6.1) can be separated into two parts, for each of which Theorem 6.2 can be applied. Thus $\{m \in \mathbf{R}^\mathbf{T} : m$ is feasible$\}$ is the intersection of two integral bi-submodular polyhedra.

Unlike the intersection of two integral polymatroids, the intersection of two integral bi-submodular polyhedra is not necesserily integral. However, it was proved by [Cunningham, 1989]to be half-integral. Note also that if each of these polyhedra is given by a membership oracle then, using the ellipsoid method, an optimal half-integral element of the intersection can be found in polynomial time. (To our best knowledge no combinatorial algorithm is known in the general case.) It follows that the following problem is polynomially solvable.

Let G and H be as in Theorem 4.3 but no Eulerian properties are required. Let $w : V \to \mathbf{Q}$ be a vector. Find a maximum weight (fractional) H-admissible multiflow, where the weight of a multiflow is defined to be $\sum(w(t)m(t) : t \in T)$ and $m(t)$ denotes the sum of values of the multiflow on the paths ending at t.

The arguments above imply that this problem has a quarter-integral optimal solution, and even a half-integral one when G is Eulerian. In addition, such a solution can be found in polynomial time (relying on the ellipsoid method). So far we did not find a method for the integral weighted maximization problem even if G is Eulerian.

Let us add that in our special case the membership (and even separation) problem for the bi-submodular polyhedron can be solved by a combinatorial strongly polynomial algorithm using a maximum flow subroutine.

References

A. Bouchet and W.H. Cunningham [1991]: Delta-matroids, jump systems and bi-submodular polyhedra, *SIAM J. on Discrete Mathematics*, to appear.

B.V. Cherkasskij [1977]: A solution of a problem of multicommodity flows in a network, *Ekon. i Mat. Metody* 13 (1) 143-151 (in Russian).

W.H. Cunningham [1989]: oral communication.

J. Edmonds [1970]: Submodular functions, matroids, and certain polyhedra in: *Combinatorial Structures and their applications* (R. Guy, H. Hanani, N. Sauer, and J. Schönheim, eds.) Gordon and Breach, New York, 69-87.

A. Frank [1990]: Packing paths, circuits, and cuts — a survey, in: *Paths, flows and VLSI-layout* (*Algorithms and Combinatorics*, vol. 9) (B. Korte, L. Lovász, H.J. Prömel, and A. Schrijver, eds.) Springer, Berlin–Heidelberg–New York–London, 47-100.

A.V. Karzanov [1984]: A generalized MFMC-property and multicommodity cut problems, in: *Finite and Infinite Sets* (Proc. 6th Hungarian Combinatorial Coll., Eger 1981) (A. Hajnal, L. Lovász, V.T. Sós, eds.), vol. 2, North-Holland, Amsterdam, 443-486.

A.V. Karzanov [1985]: On multicommodity flow problems with integer-valued optimal solutions, *Dokl. Akad. Nauk SSSR* 280 (4) 789-792 (in Russian) (English translation: *Soviet Math. Dokl.* 32 (1985) 151-154).

A.V. Karzanov [1987]: On a class of maximum multicommodity flows with integer-valued optimal solution, in: *Modelling and Optimization in Complex Systems* (G.S. Fridman, ed.) Omsk State Univ., Omsk, 103-121 (in Russian).

A.V. Karzanov and M.V. Lomonosov [1978]: Systems of flows in undirected networks, in: *Mathematical Programming etc.*, iss. 1, Inst. for Systems Studies, Moscow, 59-66 (in Russian).

A.V. Karzanov and P.A. Pevzner [1979]: A characterization of the class of cut-non-determined maximum multicommodity flow problems, in: *Combinatorial methods for flow problems*, iss. 3, Inst. for Systems Studies, Moscow, 70-81 (in Russian).

M.V. Lomonosov [1979]: Combinatorial approaches to multiflow problems, *Graph Theory Newsletters* 9, p. 4.

M.V. Lomonosov [1985]: Combinatorial approaches to multiflow problems, *Discrete Applied Mathematics* 11 (1) 1-93.

L. Lovász [1976]: On some connectivity properties of Eulerian graphs, *Acta Mat. Akad. Sci. Hungaricae* 28 129-138.

B. Rothschild and A. Whinston [1966a]: On two-commodity network flows, *Operations Research* 14 377-387.

B. Rothschild and A. Whinston [1966b]: Feasibility of two-commodity network flows, *Operations Research* 14 1121-1129.

P. Schönsleben [1980]: Ganzzahlige Polymatroid Intersections Algorithmen, *Ph.D. Thesis*, Eidgenössischen Techn. Hochschule, Zürich.

P.D. Seymour [1980]: Four-terminus flows, *Networks* 10 79-86.