**Metrics with finite sets of primitive extensions** *

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**Abstract.** This paper studies the class of rational-valued metrics $\mu$ on finite sets $T$ such that the set $\Pi(\mu)$ of primitive extensions of $\mu$ is finite. Here a positive extension $m$ of $\mu$ to a set $V \supseteq T$ is said to be *primitive* if $m$ dominates no convex combination of other extensions of $\mu$.

We show that $\Pi(\mu)$ is finite if and only if there exists a graph $H$ such that its path metric $\mathrm{d}^H$ has no primitive extensions except itself and $\lambda\mu$ is a submetric of $\mathrm{d}^H$ for some integer $\lambda > 0$. We also give a combinatorial characterization for such metrics $\mu$, explain that the finiteness of $\Pi(\mu)$ can be recognized efficiently, show that $\Pi(\mu)$ is finite if and only if the tight span of $\mu$ is 2-dimensional, demonstrate applications to multicommodity flows, and present other results. Our results rely on properties of graphs $H$ with $|\Pi(\mathrm{d}^H)| = 1$ established in [10].

*Key words*: Finite metric, Metric extension, Isometric embedding, Multi-terminal cut, Multicommodity flow, Tight span.

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## 1. Introduction

This paper continues a study of finite metrics from the viewpoint of their primitive extensions begun in [10] where necessary and sufficient conditions for a graph metric that admits a unique primitive extension are described. Here we give a complete characterization of the set of finite metrics that have possibly more than one but a finite number of primitive extensions, thus answering a question raised in [9].

Throughout by a *metric* on a set $X$ we mean a nonnegative real-valued function $d$ that establishes *distances* on the pairs of elements of $X$ satisfying (i) $d(x,x)=0$, (ii) $d(x,y)=d(y,x)$, and (iii) $d(x,y)+d(y,z)\geq d(x,z)$, for all $x,y,z\in X$. Unless otherwise is explicitly said, we assume that $X$ is finite and allow zero distances between different elements (i.e., $d$ is, in fact, a semimetric); $d$ is called *positive* if $d(x,y)>0$ for all distinct $x,y\in X$. We usually do not distinguish between the metric $d$ and metric space $(X,d)$; elements of $X$ are called *points* of this space. Because of (i) and (ii), it suffices to define $d$ on the set $E_X$ of unordered pairs of distinct elements of $X$, or, equivalently, on the edge set of the complete (undirected) graph $K_X=(X,E_X)$. We write $xy$ and $d(xy)$ in place of $\{x,y\}$ and $d(x,y)$, respectively. A special case of positive metrics is the *distance function*, or *path metric*, $\mathrm{d}^G$ of a connected graph $G=(V,E)$, where $\mathrm{d}^G(xy)$ is the minimum number of edges of a path in $G$ connecting nodes $x$ and $y$. When the edges $e$ of $G$ are endowed with nonnegative lengths $\ell(e)$, the corresponding path metric is denoted by $\mathrm{d}^{G,\ell}$ (i.e., $\mathrm{d}^{G,\ell}(xy)$ is the minimum length $\sum(\ell(e_i)\,:\,i=1,\dots,k)$ of a path $P=(x_0,e_1,x_1,\dots,e_k,x_k)$ in $G$ connecting $x=x_0$ and $y=x_k$).

We consider a positive rational-valued metric $\mu$ on a set $T$. Let $m$ be a metric on $V\supseteq T$. If $m(st)=\mu(st)$ for all $s,t\in T$, then $m$ is called an *extension* of $\mu$ to $V$, and $\mu$ a *submetric* of $m$ on $T$, denoted as $\mu=m_{|T}$. The set of extensions of $\mu$ to $V$ forms a polyhedron in $\mathbb{R}^{E_V}$, and the vertices of the dominant of this polyhedron are called *extreme* extensions. That is, an extension $m$ of $\mu$ to $V$ is extreme if $m$ dominates no convex combination of other extensions, i.e., $m\geq\lambda m'+(1-\lambda)m''$ cannot hold for any extensions $m',m''\neq m$ and real $0\leq\lambda\leq 1$. A positive extreme extension is called *primitive*; let $\Pi(\mu)$ denote the set of such extensions (regarding all finite sets $V\supseteq T$). In particular, $\mu\in\Pi(\mu)$.

Obviously, $m(xy)=0$ for some $x,y\in V$ provides that $m(xz)=m(yz)$ for all $z\in V$. This easily implies that $m$ is extreme if and only if the submetric of $m$ on $V-\{x\}$ is extreme, or, equivalently, if and only if shrinking each maximal subset of points with zero distances between them into a single point results in a primitive extension of $\mu$ to the factor set. Thus, each primitive extension represents the corresponding set of extreme extensions; the members of this set are called *similar*.

We are interested in the question: given a metric $\mu$, how large is the set $\Pi(\mu)$?

Earlier necessary and sufficient conditions have been found for the path metrics $\mathrm{d}^H$ that admit no primitive extensions except themselves.

**Theorem 1.1** [10]. *Let $H$ be a connected graph. Then $|\Pi(\mathrm{d}^H)| = 1$ if and only if $H$ is bipartite, orientable and contains no isometric $k$-cycle with $k \geq 6$.*

Here a *$k$-cycle* is a (simple) circuit $C_k$ on $k$ nodes (considered as a closed path or as a graph depending on the context). A graph $H$ is *orientable* if the edges of $H$ can be oriented so that for any 4-cycle $C = (v_0, e_1, v_1, ..., e_4, v_4 = v_0)$, the orientations of the opposite edges $e_1$ and $e_3$ are different along the cycle, and similarly for $e_2$ and $e_4$ (a feasible orientation is depicted in Fig. 2). A subgraph $H'$ of $H$ is *isometric* if $\mathrm{d}^{H'}(xy) = \mathrm{d}^H(xy)$ for any nodes $x, y$ of $H'$. The graphs $H$ as in Theorem 1.1 are called *frames*. We refer to [10] for other results on frames, in particular, those related to a generalization of the multiterminal cut problem from [6].

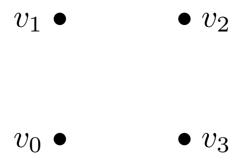


Fig. 1

CIn this paper we focus on the case when $\Pi(\mu)$ is finite. Such a metric $\mu$ is called *primitively finite*, or a *PF-metric*. (Our initial motivation for studying this case came up from the multiflow (multicommodity flow) area. Extreme extensions arise as optimal dual solutions in one sort of multiflow problems where one is asked for maximizing the sum of flow values weighted by a given metric $\mu$ on the set of terminals. The finiteness of $\Pi(\mu)$ means that, up to the similarity, the number of unavoidable optimal dual solutions occurring in the problem instances concerning this $\mu$ is finite. These aspects will be discussed in Section 6.) Our main result is the following.

**Theorem 1.2.** *Let $\mu$ be a positive rational metric on a finite set $T$. The following are equivalent:*

(i) $\Pi(\mu)$ *is finite;*

(ii) *there exist a frame $H$ and an integer $\lambda > 0$ such that $\lambda\mu$ is a submetric of* $\mathrm{d}^H$*;*

(iii) *the least generating graph $G = (V, E)$ for a modular closure $(V, m)$ of $\mu$ is bipartite and contains as an isometric subgraph neither $C_k$ with $k \geq 6$ nor $K_{3,3}^-$.*

We have to explain the notions used in (iii) of this theorem that gives a combinatorial characterization of the PF-metrics.

First, for a metric $m$ on $V$, a point $v \in V$ is called a *median* of a triple $\{s_0, s_1, s_2\}$ in $V$ if

$$m(s_i v) + m(v s_j) = m(s_i s_j) \qquad \text{for all } 0 \leq i < j \leq 2. \tag{1.1}$$

When a median exists for each triple, $m$ is called *modular*. By a *modular closure* of $\mu$ we mean a certain its extension $(V, m)$ which is modular and is constructed by the following process. Initially set $V := T$ and $m := \mu$. Choose in $V$ a triple $\{s_0, s_1, s_2\}$ without a median, add a new point $v$ to $V$ and define the distances from $v$ to the $s_i$'s so as to satisfy (1.1) (such distances exist and are unique). Then define distances from $v$ to the other points in $V$ as follows. Let $V' \subset V$ be the set of points of which distances from $v$ have already been defined; initially $V' = \{s_0, s_1, s_2, v\}$. Choose a point $u \in V - V'$ and put

$$m(uv) := \max\{m(ux) - m(xv) \,:\, x \in V' - \{v\}\}. \tag{1.2}$$

Update $V' := V' \cup \{u\}$ and iterate until $V' = V$. One can see that $m$ remains an extension of $\mu$. Repeat this procedure for a next medianless triple in the current $(V, m)$, and so on.

When the process terminates, the resulting $(V, m)$ has medians for all triples and is just the desired extension of $\mu$. Note that a priori $m$ depends on the order in which the medianless triples are treated (however, it is invariant when $\mu$ is primitively finite, as we explain in Section 5). We show in Section 2 that for a rational metric $\mu$ the above process does terminate in a finite number of steps and, moreover, that the resulting $m$ is a primitive extension of $\mu$.

Second, a spanning subgraph $G$ of $K_V$ is said to *generate* $m$ if $m$ coincides with the path metric defined by $G$ whose each edge $e$ has length $m(e)$. The *least graph generating*, or, briefly, *LG-graph*, of $m$ is obtained by deleting all redundant edges from $K_V$, where $xy$ is *redundant* if there is $z \in V - \{x, y\}$ with $m(xz) + m(zy) = m(xy)$ (such a $z$ is said to be *between* $x$ and $y$).

Third, $K^-_{3,3}$ is the graph obtained by deleting one edge from $K_{3,3}$, where $K_{p,q}$ is the complete bipartite graph whose parts (i.e., the maximal stable sets) consist of $p$ and $q$ nodes; see Fig. 2. We call a bipartite graph without isometric subgraphs $C_k$, $k \geq 6$, and $K^-_{3,3}$ a *semiframe*. Note that $K^-_{3,3}$ is non-orientable, so every frame is a semiframe. On the other hand, every graph $K_{p,q}$ is a semiframe, but it is not a frame if $p, q \geq 3$. In a bipartite graph an induced subgraph $C_6$ or $K^-_{3,3}$ is, obviously, isometric.

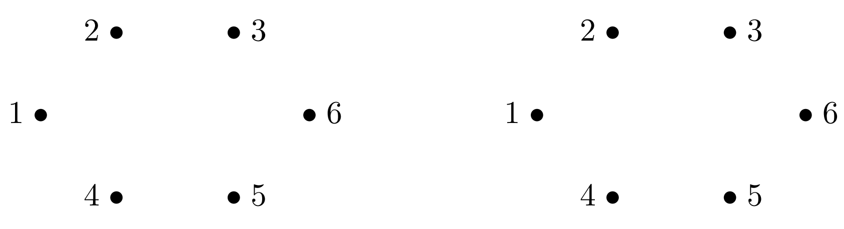


Fig. 2 (a) $K_{3,3}$ (b) $K^-_{3,3}$

We now briefly outline the method of proof of Theorem 1.2. Implication (ii)$\rightarrow$(i) will follow from Theorem 1.1 and a rather simple fact that if $\mu$ is a submetric of a positive metric $d$, then every primitive extension of $\mu$ corresponds to a submetric of a primitive extension of $d$. So $|\Pi(\mu)|$ does not exceed the number of submetrics of $\mathrm{d}^H$ and is, therefore, finite. The proof of implication (i)→(iii) relies on the following result.

**Theorem 1.3** [10]. *If a connected graph $H$ contains a 6-cycle or $K_{3,3}^-$ as an isometric subgraph, then there exists a primitive extension $m'$ of $\mathrm{d}^H$ which has a submetric $m''$ on six points such that $m'' = \frac{1}{2}\mathrm{d}^{K_{3,3}^-}$.*

This will enable us to recursively construct an infinite sequence of primitive extensions of $\mu$ in the case when $G = (V, E)$ as above is not a semiframe. Implication (iii)→(ii) is more complicated to prove. To this aim, we elaborate a so-called *orbit splitting method* that transforms, step by step, the semiframe $G$ figured in (iii) into the desired frame $H$ (with $\lambda\mu$ to be a submetric of $\mathrm{d}^H$).

Next, obviously, for any $\lambda > 0$, $m' \in \Pi(\mu)$ if and only if $\lambda m' \in \Pi(\lambda\mu)$. Therefore, without affecting our problem one may assume that $\mu$ is integer-valued (as we deal with a rational metric on $T$). For our purposes it is more convenient to assume that $\mu$ is *cyclically even*. This means that $\mu(xy) + \mu(yz) + \mu(zx)$ is an even integer for all $x, y, z \in T$ (in particular, $\mu(xy)$ is an integer for all $x, y \in T$ since $\mu(xy)+\mu(yy)+\mu(yx)$ is even). Then a modular closure $m$ of such a $\mu$ is cyclically even as well, as explained in Section 2. This property is used in our construction of the desired frame $H$ in the proof of (iii)→(ii). Moreover, the factor $\lambda$ in the embedding $\lambda\mu \to \mathrm{d}^H$ turns out to be 1 or 2. As a consequence, we obtain the following result on the fractionality of primitive extensions of PF-metrics.

**Theorem 1.4.** *If $\mu$ is a cyclically even PF-metric, then every primitive extension of $\mu$ is half-integral. If, in addition, the LG-graph of a modular closure of $\mu$ is a frame, then every primitive extension of $\mu$ is integral.*

The next result concerns tight spans of metrics (also known in literature as injective envelopes or $T_X$-spaces). For a positive metric $d$ on a (not necessarily finite) set $X$, an extension $d'$ of $d$ to $X' \supseteq X$ is called *tight* if $d'$ dominates no other extension $d''$ of $d$ to $X'$ (i.e. $d'' \leq d'$ is impossible). The *tight span* of $d$ is its positive tight extension $\mathcal{T}(d) = (\mathcal{X}, \delta)$ such that every tight extension $(X', d')$ of $d$ is isometrically embeddable in $(\mathcal{X}, \delta)$, in the sense that there exists a mapping $\gamma : X' \to \mathcal{X}$ satisfying $\gamma(x) = x$ for all $x \in X$ and $\delta(\gamma(x)\gamma(y)) = d'(xy)$ for all $x, y \in X'$. Isbell [8] and Dress [7] proved the existence and uniqueness of such an $(\mathcal{X}, \delta)$ for any metric space $(X, d)$. Moreover, when $X$ is finite, $\mathcal{X}$ can be represented as a polyhedral complex whose dimension is shown to be at most $|X|/2$ (formally, for $X$ finite, let $Q$ be the set of nonnegative vectors $\xi \in \mathbb{R}^X$ satisfying $\xi_x + \xi_y \geq d(xy)$ for all $x, y \in X$; then $\mathcal{X}$ is the part of (the boundary of) $Q$ formed by the vectors $\xi \in Q$ dominating no vector in $Q - \{\xi\}$, each point $x \in X$ is identified with the vector $\xi$ such that $\xi_y = d(xy)$ for all $y \in X$, and $\delta$ is the $\ell_\infty$-metric, i.e., $\delta(\xi\xi') = \max\{|\xi_x - \xi'_x| \,:\, x \in X\}$).

It is shown in [10] that $\mathcal{T}(\mathrm{d}^H)$ is at most 2-dimensional for any frame $H$ (by giving an explicit combinatorial construction of $\mathcal{T}(\mathrm{d}^H)$). We generalize this result by completely characterizing the set of 2-dimensional tight spans of rational finite metrics (it is known that the 1-dimensional tight spaces arise from tree-wise metrics [7]).

**Theorem 1.5.** *$\Pi(\mu)$ is finite if and only if the tight span of $\mu$ has dimension at most two. Moreover, for any PF-metric $\mu$, $\mathcal{T}(\mu)$ is isomorphic to $\mathcal{T}(\mathrm{d}^H/\lambda)$ for some frame $H$ and integer $\lambda > 0$. In other words, up to proportionality, the 2-dimensional tight spans of rational finite metrics are exactly the tight spans of the path metrics of frames (different from trees).*

Next, Dress pointed out two important local properties of metrics.

**Theorem 1.6** [7]**.** *For a metric space $(X, d)$,*

(i) *if $\mathcal{T}(d)$ is $k$-dimensional ($k < \infty$), then there is a submetric $d'$ of $d$ on $2k$-points such that $\mathcal{T}(d')$ is $k$-dimensional;*

(ii) *if $|X| = 2k$, then $\mathcal{T}(d)$ is $k$-dimensional if and only if there exists a bijection (i.e., a one-to-one mapping) $\pi : X \to X$ satisfying $\pi(v) \neq v$ and $\pi(\pi(v)) = v$ for all $v \in X$ (i.e., $\pi$ is an involution) and such that $\sum(m(v\pi(v)) \,:\, v \in X) > \sum(m(v\pi'(v)) \,:\, v \in X)$ holds for any other bijection $\pi' : X \to X$.*

Statement (i) of this theorem and Theorem 1.5 provide the following characterization of PF-metrics in local terms.

**Corollary 1.7.** *$\mu$ is primitively finite if and only if each submetric of $\mu$ on six points is so.*

In its turn, (ii) in Theorem 1.6 together with Theorem 1.5 shows that the problem of deciding whether $\Pi(\mu)$ is finite or not is solvable in strongly polynomial time. Indeed, we can simply enumerate all six-element subsets $T'$ of $T$, and for each such $T'$, enumerate all bijections $T' \to T'$ and verify the corresponding inequalities.

This paper is organized as follows. Section 2 justifies the process of constructing a modular closure $m$ of $\mu$ and exhibits basic properties of $m$ and its LG-graph $G$. Section 3 proves (ii)$\to$(i) and (i)$\to$(iii) in Theorem 1.2. Section 4 describes the orbit splitting method and applies it to prove (iii)$\to$(ii) in Theorem 1.2. Section 5 explains how Theorems 1.4 and 1.5 follow and gives some additional results. In particular, we show there that for any PF-metric $\mu$, the modular closure of $\mu$ is determined uniquely, and there is a canonical pair $(H, h)$, where $H$ is a frame and $h$ is a length function on its edges, such that $\mathrm{d}^{H,h}$ is a primitive extension of $\mu$ and every primitive extension of $\mu$ is isomorphic to a submetric of $\mathrm{d}^{H,h}$. The concluding Section 6 demonstrates applications to multiflows.

Among a variety of tools used in our proofs, we, in particular, apply results of Bandelt on hereditary modular graphs. A graph $H$ is called *modular* if $\mathrm{d}^H$ is modular, and *hereditary modular* if each isometric subgraph of $H$ is modular. In particular, any modular graph is bipartite.

**Theorem 1.8** [3]**.** *Let $H = (T, U)$ be a graph.*

(i) *$H$ is hereditary modular if and only if $H$ is bipartite and contains no isometric $k$-cycle with $k \geq 6$.*

(ii) *If $H$ is modular but not hereditary modular, then $H$ contains an isometric 6-cycle that, in its turn, is contained in a (not necessarily induced) cube in $H$ (see Fig. 3).*

(iii) *If $H$ is bipartite but not modular, then $H$ contains a medianless triple $\{s_0, s_1, s_2\}$ with $\mathrm{d}^H(s_0s_1) = \mathrm{d}^H(s_0s_2) \geq 2$ and $\mathrm{d}^H(s_1s_2) = 2$.*

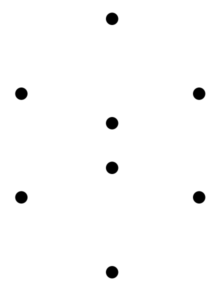

Fig. 3

In view of (i), the frames (semiframes) are exactly those hereditary modular graphs which are orientable (respectively, without induced subgraphs $K^-_{3,3}$).

A majority of results in this paper can be extended, with a due care, to arbitrary, not necessarily rational and finite, metrics $\mu$; we, however, consider only the finite rational case to make our description shorter and technically simpler.

## 2. The modular closure and least generating graph

Let $m$ be an extension of $\mu$ to $V$. A sequence $P = (x_0, x_1, \ldots, x_k)$ of points of $V$ is called a *path on* $V$; we refer to $P$ as a *$T$-path* if $x_0, x_k \in T$, and a *cycle* if $x_0 = x_k$. For brevity we write $P = x_0x_1 \ldots x_k$. The *length* of $P$ with respect to $m$, or the *$m$-length*, is $m(P) = m(x_0x_1) + \ldots + m(x_{k-1}x_k)$, and $P$ is called *$m$-shortest* if $m(P) = m(x_0x_k)$. The set of $m$-shortest $T$-paths is denoted by $\mathcal{G}(m) = \mathcal{G}(T, m)$.

If extensions $m, m', m''$ of $\mu$ to $V$ satisfy $m \geq \lambda m' + (1 - \lambda)m''$ with $0 < \lambda \leq 1$, we say that $m'$ *decomposes* $m$. So $m$ is extreme if and only if no $m' \neq m$ decomposes $m$.

It is easy to see that $m'$ decomposes $m$ if and only if $\mathcal{G}(m) \subseteq \mathcal{G}(m')$. A practical use of this property is that if we can show, by one or another method, that $m$ is determined uniquely by $\mu$ and the set of $m$-shortest $T$-paths and, then we can declare that $m$ is extreme (or primitive when $m$ is positive).

In what follows we assume that $\mu$ is cyclically even. First of all we show that the construction of a modular closure $(V, m)$ for $\mu$ described in the Introduction is well-defined. Here and later on we need a simple characterization of tight extensions in terms of shortest paths (see, e.g., [8]), namely: an extension $m'$ of $\mu$ to $V'$ is tight if and only if

(2.1) for any $x, y \in V'$, there are $s, t \in T$ such that $m'(sx) + m'(xy) + m'(yt) = m'(st)$ $(= \mu(st))$, i.e., $x, y$ are contained in an $m'$-shortest $T$-path.

**Statement 2.1.** *The process of constructing a modular closure $(V, m)$ terminates in a finite number of iterations. Moreover, $m$ is cyclically even and primitive.*

*Proof.* Suppose that after a number of iterations we have obtained a cyclically even primitive extension $m$ on a current set $V$, and let the next iteration add a median $v$ for a triple $\{s_0, s_1, s_2\}$. By (1.1), $m(s_0v)$ is uniquely determined to be $\frac{1}{2}(m(s_0s_1) + m(s_0s_2) - m(s_1s_2))$, and similarly for $m(s_1v)$ and $m(s_2v)$. So the numbers $m(s_iv)$ are positive integers. Moreover, the submetric of $m$ on $V^0 = \{s_0, s_1, s_2, v\}$ is, obviously, cyclically even.

Let $V - V^0$ consist of points $s_3, \ldots, s_n$ which are chosen in this order when the distances from $v$ to these points are determined. By rule (1.2), for each $i = 3, \ldots, n$, there is $j < i$ such that $m(s_iv) + m(vs_j) = m(s_is_j)$, i.e., the path $P_i = s_ivs_j$ is $m$-shortest. Then (by induction on $i$) $m(s_iv)$ is an integer and the $m$-length of the cycle $s_ivs_js_i$ is even. This easily implies that the $m$-length of any cycle of the form $s_pvs_qs_p$, $p, q = 1, \ldots, n$ is even, and now the fact that the metric on $V := \{s_0, \ldots, s_n, v\}$ is cyclically even follows from a similar property for its submetric $\widetilde{m}$ on $\widetilde{V} = \{s_0, \ldots, s_n\}$. Also the new $m$ is positive (for if $m(vs_i) = 0$ for some $i$, then $s_i$ is a median for $\{s_0, s_1, s_2\}$).

Assume by induction that the previous metric $\widetilde{m}$ is primitive. To see that the new $m$ is primitive, consider the paths $P_i$, $i = 3, \ldots, n$ as above and the paths $P_0 = s_0vs_2$, $P_1 = s_1vs_0$ and $P_2 = s_2vs_1$. Since $\widetilde{m}$ is primitive and, therefore, tight, the ends $s_i$ and $s_j$ of each $P_i$ are contained in an $\widetilde{m}$-shortest $T$-path $ss_is_jt$ (cf. (2.1)). Then the $T$-path $P'_i = ss_ivs_jt$ is $m$-shortest. Suppose that an extension $m'$ of $\mu$ to $V$ decompose $m$. Since $\widetilde{m}$ is primitive, $m(xy) = m'(xy) = \widetilde{m}(xy)$ for all $x, y \in \widetilde{V}$. Also $\mathcal{G}(m) \subseteq \mathcal{G}(m')$ implies that each $P'_i$ is $m'$-shortest. Then each $P_i$ is $m'$-shortest. But the system $\{P_0, \ldots, P_n\}$ of shortest paths determine the distances on $vs_0, \ldots, vs_n$ uniquely; so

$m(vs_i) = m'(vs_i)$, $i = 0, \ldots, n$. Therefore, $m' = m$, yielding the primitivity of $m$.

To estimate the number of iterations in the process, associate with each point $x$ of the current $(V, m)$ the vector $\xi(x) = (m(xs) : s \in T)$. Since $m$ is tight and positive, (2.1) implies that $m(sx) \le \max\{\mu(s't') : s', t' \in T\} =: a$ and that $\xi(x) \ne \xi(y)$ for any distinct $x, y \in V$. Thus, $|V| < a^{|T|}$, and the process is finite. •

In the proof of Theorem 1.2 given in next sections we will essentially use the fact that the shortest paths of a modular closure of $\mu$ are closely related to the shortest paths of its LG-graph; such a property was established by Bandelt for arbirtary modular metric spaces. For a graph $G = (V, E)$, a path $P = x_0x_1 \ldots x_k$ on $V$ is a *path in* $G$ if $x_{i-1}x_i \in E$ for $i = 1, \ldots, k$. If $x_0 = x_k$ and all edges $x_{i-1}x_i$ are different, $P$ is a *cycle in* $G$. The number $k$ of edges of $P$ is denoted by $|P|$ and called the *G-length* of $P$. A shortest path in $G$ is called *G-shortest*. An *s–t path* is a path with ends $s, t$.

**Statement 2.2** [2] (see also [4]). *Let $m$ be a positive modular metric on a set $V$, and $G = (V, E)$ its LG-graph. Then a path in $G$ is $m$-shortest if and only if it is $G$-shortest. Therefore, the sets of $m$-shortest and $\mathrm{d}^G$-shortest paths on $V$ are the same.*

To make our description more self-contained, we give a proof of this statement.

*Proof.* First we observe that

(2.2) every simple path $P'$ with $|P'| = 2$ in $G$ is simultaneously $m$-shortest and $G$-shortest.

Indeed, let $P' = xyz$. Take a median $v$ w.r.t. $m$ for $\{x, y, z\}$. If $v = y$ then $m(xy) + m(yz) = m(xz)$. So $P'$ is $m$-shortest, whence $P'$ is $G$-shortest (since $y$ is between $x$ and $y$ for $m$). And if $v \ne y$ then, letting for definiteness that $v \ne x$, the equality $m(xv) + m(vy) = m(xy)$ shows that the edge $xy$ is redundant in $G$; a contradiction.

Consider two paths $P = sx_1 \ldots x_k t$ and $Q = sy_1 \ldots y_q t$ in $G$ with the same ends such that $P$ is $G$-shortest and $Q$ is $m$-shortest. It suffices to show that $m(P) = m(Q)$ and $|P| = |Q|$. We use induction on $|P|$. Case $|P| = 1$ is obvious.

(i) Let $|P| = 2$, i.e., $k = 1$. By (2.2), $m(P) = m(Q)$. Suppose that $|Q| \ge 3$. Since no edge in $G$ is redundant, each of $y_1, \ldots, y_q$ is different from $x = x_1$. Take a median $v$ for $\{s, x, y_q\}$. If $v \ne s, x$ then the edge $sx$ is redundant, while if $v = x$ then the edge $xt$ is redundant (since $Q$ and $svy_q$ are $m$-shortest, whence $m(vy_q) + m(y_q t) = m(vt)$). Therefore, $v = s$. This implies that the path $xsy_1 \ldots y_q$ is $m$-shortest, whence $m(xy_q) > m(xy_1)$ (as $q > 1$ and $m$ is positive). Arguing similarly for the triple $\{t, x, y_1\}$, we obtain the reverse inequality $m(xy_1) > m(xy_q)$; a contradiction. Thus, $|Q| = 2$.

(ii) Let $|P| \geq 3$ (then $|Q| \geq 3$). Take a median $v$ for $\{s, x_k, y_q\}$. We show that it suffices to consider the case when

(2.3) $v = s$ and the path $D = x_k \dots x_1 s y_1 \dots y_q$ is $m$-shortest.

Indeed, choose in $G$ an $m$-shortest $s$–$x_k$ path $R$ containing $v$. Since $P' = sx_1 \dots x_k$ is $G$-shortest and $|P'| < |P|$, we have by induction that $|P'| = |R|$ and $m(P') = m(R)$. So we may assume that $P$ is chosen so that $P' = R$; then $v = x_i$ for some $i$ (letting $x_0 = s$). Case $i = 0$ gives (2.3) (in view of $m(x_k v) + m(v y_q) = m(x_k y_q)$). Let $i > 0$. Consider the concatenation $L$ of the path $sx_1 \dots x_i$, an $m$-shortest $x_i$–$y_q$ path, and the path $y_q t$. One can see that $L$ is $m$-shortest. Let $L'$ and $L''$ be the parts of $L$ from $s$ to $y_q$ and from $x_i$ to $t$, respectively. By induction the paths $P'' = x_i \dots x_k t$ and $L''$ satisfy $|P''| = |L''|$ and $m(P'') = m(L'')$, whence $P$ is $m$-shortest and $L$ is $G$-shortest. Again applying induction to $L'$ and $Q' = sy_1 \dots y_q$, we have $|L'| = |Q'|$, which implies that $Q$ is $G$-shortest.

Finally, consider $D$ in (2.3) and $\widetilde{P} = x_k t y_q$. By (2.2), $\widetilde{P}$ is $G$-shortest. Then $|D| = |\widetilde{P}| = 2$, by (i). Therefore, $D = x_1 s y_1$ (and $k = q = 1$). This gives $|P| = |Q| = 2$, and the result follows. •

This statement provides that $G$ is modular. Indeed, for any $s_0, s_1, s_2 \in V$, there are $s_0$–$s_1$, $s_1$–$s_2$ and $s_2$–$s_0$ paths in $G$ which are $m$-shortest and share a common node $v$. Then these paths are $G$-shortest, therefore, $v$ is a median for $\{s_0, s_1, s_2\}$ w.r.t. $\mathrm{d}^G$. Another corollary from Statement 2.2 is that every isometric subgraph (or cycle) $G'$ in $G$ is *$m$-isometric*, i.e., any two nodes in $G'$ are connected by an $m$-shortest path which is entirely contained in $G'$. Moreover, in view of (2.1), the tightness of $m$ enables us to sharpen this property as follows:

(2.4) if $G' = (V', E')$ is an isometric subgraph (or cycle) in $G$, then any two $x, y \in V'$ belong to an $m$-shortest $T$-path in $G$ whose part between these nodes is contained in $G'$.

## 3. Proof of (ii)→(i)→(iii) in Theorem 1.2

We need two simple facts about extreme extensions.

**Statement 3.1** [12]. (i) *If $(T', \mu')$ is a submetric of $(V', m')$, and $(V'', m'')$ is an extreme extension of $\mu'$ with $V' \cap V'' = T'$, then there exists an extreme extension $\widetilde{m}$ of $m'$ to $W = V' \cup V''$ such that $\widetilde{m}$ coincides with $m''$ on $V''$.*

(ii) *If $(V_1, m_1)$ is an extreme extension of $(V_2, m_2)$ which, in its turn, is an extreme* *extension of $(V_3, m_3)$, then $m_1$ is extreme for $m_3$.*

(To see (i), define $d(xy)$ to be $m'(xy)$ for $x, y \in V'$, $m''(xy)$ for $x, y \in V''$, and $\min\{m'(xs) + m''(sy) : s \in T'\}$ for $x \in V'$ and $y \in V''$. One can check that $d$ is a metric on $W$ and, therefore, an extension of $m'$. Let $\widetilde{m}$ be an extreme extension of $m'$ to $W$ which decomposes $d$. Then $\widetilde{m}_{|V''}$ is an extension of $\mu'$ which decomposes $m''$. Since $m''$ is extreme for $\mu'$, we have $\widetilde{m}_{|V''} = m''$; so $\widetilde{m}$ is as required. To see (ii), consider an extreme extension $q$ of $m_3$ which decomposes $m_1$; then $q' = q_{|V_2}$ decomposes $m_2$ , whence $q' = m_2$, implying $q = m_1$.)

Now we can prove (ii)→(i) in Theorem 1.2 as follows. Let a metric $\mu$ on $T$ be such that $\mu' = \lambda\mu$ is a submetric of $m' = \mathrm{d}^H$ for a frame $H = (V', E')$ and number $\lambda > 0$. Consider a primitive extension $(V'', m'')$ of $\mu'$, assuming $V' \cap V'' = T$. Take an extreme extension $\widetilde{m}$ of $m'$ to $W = V' \cup V''$ with $\widetilde{m}_{|V''} = m''$, existing by (i) in Statement 3.1. By Theorem 1.1, $m'$ has only one primitive extension, namely, $m'$ itself. Therefore, $\widetilde{m}$ and $m'$ are similar, which means that each point $v \in W$ is at zero $\widetilde{m}$-distance from some point $v' = \gamma(v)$ of $V'$. Then $m''(uv) = m'(\gamma(u)\gamma(v))$ for any $u, v \in V''$, i.e., $m''$ is isomorphic to the submetric of $m'$ on the set $\gamma(V'')$. Thus, the number of primitive extensions of $\mu$ does not exceed the number $2^{V'}$ of submetrics of $\mathrm{d}^H$, whence $\Pi(\mu)$ is finite.

Next we prove (i)→(iii) in Theorem 1.2. Let the LG-graph $G = (V, E)$ of a modular closure $m$ of $\mu$ is not a semiframe. We recursively construct an infinite sequence of different primitive extensions of $\mu$, using Theorem 1.3 and Statement 3.1.

As mentioned in Section 2, $G$ is modular. In particular, $G$ is bipartite and, therefore, any 4-cycle in $G$ is isometric. Consider an isometric cycle $C = v_0 v_1 \ldots v_{2k-1} v_0$ in $G$. By (2.4) (with $G' = C$), for each $i = 0, \ldots, k-1$, the opposite nodes $v_i$ and $v_{i+k}$ belong to a shortest $T$-path whose part $P$ between these nodes is of the form $v_i v_{i+1} \ldots v_{i+k}$, taking indices modulo $2k$. Then

$$\sum(m(v_{i+q} v_{i+q+1}) : q = 0, \ldots, k-1) = \sum(m(v_{i+q} v_{i+q+1}) : q = k, \ldots, 2k-1),$$

Comparing two such equalities for $i = j, j+1$ shows that the distances of any two opposite edges in $C$ are the same,

$$m(v_j v_{j+1}) = m(v_{j+k} v_{j+k+1}) \qquad \text{for } j = 0, \ldots, k-1. \tag{3.1}$$

Avis [1] and Lomonosov [14] used isometric even cycles and relations of the form (3.1) to prove irreducibility of the metrics of certain graphs. We apply a similar approach to construct the desired primitive extensions in our case. Edges $e, e'$ of $G$ are called *dependent* if there is a sequence $e = e_0, e_1, \ldots, e_p = e'$ in which each two consecutive edges $e_j, e_{j+1}$ are opposite in some (even) isometric cycle of $G$. The dependency relation is symmetric and transitive, and we call a maximal set of dependent edges an *orbit* of $G$. Then (3.1) implies that

(3.2) the distances $m(e)$ of all edges $e$ of an orbit of $G$ are the same.

Now suppose that $G$ contains an induced subgraph $H' = (T', U')$ isomorphic to $K^-_{3,3}$ (notation $H' \simeq K^-_{3,3}$). Note that $H'$ is isometric since $G$ is bipartite. Moreover, it is easy to check that all edges of $H'$ are dependent (via 4-cycles in $H'$), therefore, by (3.2),

(3.3) the submetric $\mu'$ of $m$ to $T'$ is $\lambda \mathrm{d}^{H'}$ for some $\lambda > 0$.

A feature of $K^-_{3,3}$ is that its metric $d = \mathrm{d}^{K^-_{3,3}}$ has a primitive extension which, in its turn, has a proper submetric $d'$ isomorphic to $\frac{1}{2}d$. Then $d'$ can be extended in a similar way, and one can repeat such a procedure as many times as one wishes, every time obtaining a new primitive extension of the initial metric, due to (i) and (ii) in Statement 3.1. More precisely, the following is true (cf. Theorem 1.3).

**Statement 3.2** [10]. *For $H' = (T', U') \simeq K^-_{3,3}$, there exists a bipartite graph $G'' = (V'', E'')$ with $V'' \supset T'$ such that: (i) $m'' = \frac{1}{2}\mathrm{d}^{G''}$ is a primitive extension of $\mathrm{d}^{H'}$, and (ii) $G''$ contains $K^-_{3,3}$ as an induced subgraph.*

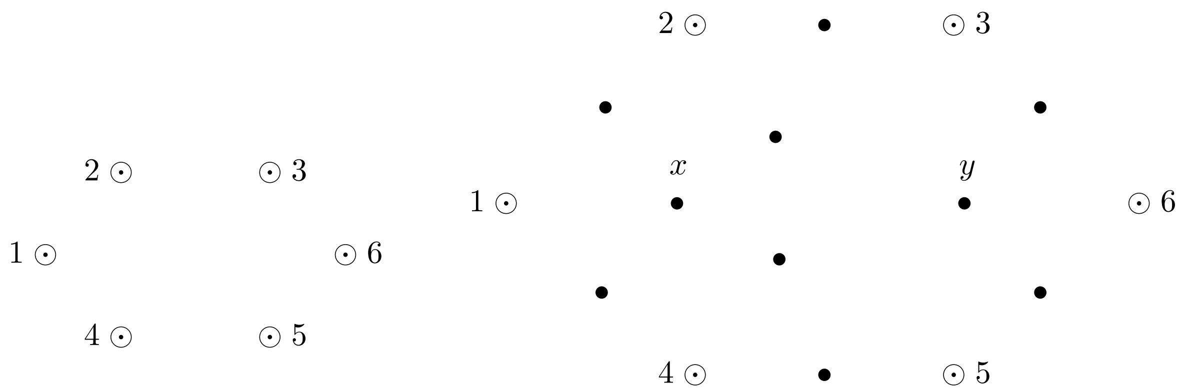


Fig. 4 (a) $H' \simeq K^-_{3,3}$ (b) $G''$

The desired graph $G''$ is drawn in Fig. 4b where for convenience the nodes of $H'$ are labelled by $1, \ldots, 6$ as indicated in Fig. 4a. This $G''$ is obtained by splitting each edge $e = ij$ of $H'$ into two edges $iz_e$ and $z_e j$ in series, and adding: (a) two extra nodes $x$ and $y$, (b) edges $xz_e$ for all $e = ij \in U'$ with $i, j \leq 5$, and (c) edges $yz_e$ for all $e = ij \in U'$ with $i, j \geq 2$. An induced subgraph $K^-_{3,3}$ in $G''$ is drawn bold in Fig. 4b. It is not difficult to check that $m''$ is an extension of $\mathrm{d}^{H'}$ and, moreover, that $m''$ is a tight extension of $\mathrm{d}^{H'}$ (e.g., $x$ and $y$ belong to a shortest path of length six in $G''$ which connects nodes 1 and 4). However, a direct verification of the primitivity of $m''$

would take some efforts. Instead, we can observe that all edges of $G''$ are dependent (via 4-cycles) and use the following fact (a sharper version of which occurs in [10]); we will use this fact once again later.

**Statement 3.3.** *Let $\mu'$ be a metric on a set $T'$, and let $G'' = (V'', E'')$ be a graph with $V'' \supseteq T'$ such that: (i) for some $\alpha > 0$, $m'' = \alpha \mathrm{d}^{G''}$ is a tight extension of $\mu'$, and (ii) all edges of $G''$ are dependent. Then $m''$ is a primitive extension of $\mu'$.*

*Proof.* By (i) and (2.4), any two opposite nodes $x, y$ in an even isometric cycle $C$ of $G''$ belong to some shortest $T'$-path whose part between $x$ and $y$ is contained in $C$. This implies that any extension $\widetilde{m}$ of $\mu'$ to $V''$ such that any shortest $T'$-path of $G''$ is $\widetilde{m}$-shortest satisfies (3.2) (with $m = \widetilde{m}$ and $G = G''$), by the argument above. Then $\widetilde{m}(e)$ is a constant $\beta$ for all $e \in E''$, by (ii). Since $m''$ is tight, any two nodes $u, v \in V''$ belong to a shortest $T'$-path $P$ in $G''$. This implies $\widetilde{m}(uv) = \beta|P'| = (\mu'(st)/|P|)|P'|$, where $s, t$ are the ends of $P$ and $P'$ is the subpath of $P$ between $u$ and $v$. Hence, $\widetilde{m}(uv) = m''(uv)$ for all $u, v \in V''$, i.e., $m''$ is primitive. $\bullet$

Formally, the process is as follows. Take the primitive extension $\lambda m''$ of $\mu'$, where $\mu', \lambda$ are as in (3.3) and $m''$ is as in Statement 3.2. By (i) and (ii) in Statement 3.1, there exists an extreme extension of $\mu$ to $V \cup V''$ which coincides with $\lambda m''$ on $V''$. This gives a primitive extension of $\mu$ which has a submetric $\mu''$ isomorphic to $\frac{1}{2}\lambda \mathrm{d}^{K^-_{3,3}}$. Then we extend $\mu''$ in a similar way, using again the construction involving a copy of the graph $G''$. This results in a new primitive extension of $\mu$ with a submetric of the form $\frac{1}{4}\lambda \mathrm{d}^{K^-_{3,3}}$. Continuing this process, we obtain an infinite sequence of different primitive extensions of $\mu$, as required.

Next we suppose that $G$ is not hereditary modular. Since $G$ is modular, $G$ contains an isometric 6-cycle $C = v_0 v_1 \dots v_5 v_0$, by (ii) in Theorem 1.8. By (3.1),

$$m(v_0v_1) = m(v_3v_4) =: \alpha, \quad m(v_1v_2) = m(v_4v_5) =: \beta, \quad m(v_2v_3) = m(v_5v_0) =: \gamma.$$

Let $\widetilde{\mu}$ be the submetric of $m$ on $\widetilde{T} = \{v_0, \dots, v_5\}$. Our goal is to find a primitive extension $\widetilde{m}$ of $\widetilde{\mu}$ such that $\widetilde{m}$ has a submetric $\mu'$ of the form $\alpha \mathrm{d}^{K^-_{3,3}}$. Then one can apply to $\mu'$ the above construction which provides infinitely many primitive extensions for the initial $\mu$.

The desired $\widetilde{m}$ is easy to construct when $\alpha = \beta = \gamma$. More precisely, let $u_i = v_i$, $i = 0, \dots, 5$, and let $\Gamma$ be the graph on eight nodes $u_0, \dots, u_5, x, y$ drawn in Fig. 5a. One can see that $\mathrm{d}^\Gamma$ is a tight extension of $\mathrm{d}^C$ and that the edges of $\Gamma$ are dependent. Therefore, $\mathrm{d}^\Gamma$ is a primitive extension of $\mathrm{d}^C$, whence there is an extreme extension of $\mu$ to $V \cup \{x, y\}$ whose submetric on the node set of $\Gamma$ is $\alpha \mathrm{d}^\Gamma$. Since $\Gamma$ contains $K^-_{3,3}$ as an induced subgraph (drawn in bold in Fig. 5a), the result follows.

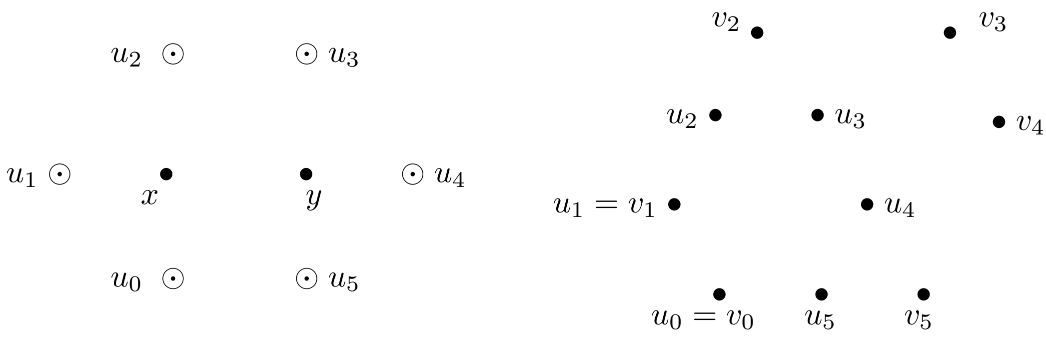


Fig. 5 (a) $\Gamma$ (b)

Now assume that the $\alpha, \beta, \gamma$ are not the same, $\alpha < \beta \leq \gamma$ say. Let $\rho$ be the distance function for a set $W$ of ten points $u_0 = v_0, u_1 = v_1, v_2, \ldots, v_5, u_2, \ldots, u_5$, defined by

$$
\begin{aligned}
\rho(v_i v_j) &= m(v_i v_j), \\
\rho(u_i u_j) &= \alpha\varphi(i,j), \\
\rho(u_i v_j) &= m(v_0 v_j) - \alpha\varphi(0,j) + \alpha\varphi(i,j)
\end{aligned} \tag{3.4}
$$

for $i, j = 0, \ldots, 5$, where $\varphi(i', j') = \mathrm{d}^C(v_{i'} v_{j'})$. One can check that $\rho$ is indeed a metric on $W$ (Fig. 5b illustrates the LG-graph for $\rho$). Moreover, by (3.4), for each $i = 0, \ldots, 5$, the $\rho$-length of the path $P_i = v_i u_i u_{i+1} u_{i+2} u_{i+3} v_{i+3}$ is equal to

$$
\begin{aligned}
&\rho(v_i u_i) + \rho(u_i u_{i+3}) + \rho(u_{i+3} v_{i+3}) \\
&\qquad = (m(v_0 v_i) - \alpha\varphi(0,i)) + 3\alpha + (m(v_0 v_{i+3}) - \alpha\varphi(0, i+3)) \\
&\qquad\qquad = m(v_i v_{i+3}) + 3\alpha - \alpha\varphi(i, i+3) = m(v_i v_{i+3}) \quad (= \alpha + \beta + \gamma),
\end{aligned}
$$

taking indices modulo 6 and using the facts that $m(v_0 v_i) + m(v_0 v_{i+3}) = m(v_i v_{i+3})$ and $\varphi(0,i) + \varphi(0, i+3) = 3$. So $P_i$ is a $\rho$-shortest $\widetilde{T}$-path. Since each two points of $W$ occur in some $P_i$, $\rho$ is a tight extension of $\widetilde{\mu}$. But $\rho$ is not primitive for $\widetilde{\mu}$. Nevertheless, we can use the fact that the submetric $\rho'$ of $\rho$ on $\{u_0, \ldots, u_5\}$ is $\alpha \mathrm{d}^{C_6}$. We further extend $\rho$ by use of the metric $\alpha \mathrm{d}^\Gamma$ (which first extends $\rho'$), where $\Gamma$ is the above-mentioned graph depicted in Fig. 5a. Then the resulting metric $\widetilde{\rho}$ on $W \cup \{x, y\}$ is already a primitive extension of $\widetilde{\mu}$. This follows from the observation that the collection of the above $\rho$-shortest paths $P_i$ together with the shortest paths between $u_i$ and $u_{i+3}$ in $\Gamma$, $i = 0, 1, 2$, determines $\widetilde{\rho}$ uniquely. Again, $\mu$ has a primitive extension in which some submetric is proportional to $\mathrm{d}^{\bar{K}_{3,3}}$, providing the existence of infinitely many primitive extensions for $\mu$. This completes the proof of implication (i)$\to$(iii) in Theorem 1.2.

## 4. Proof of (iii)$\to$(ii) in Theorem 1.2

Let the graph $G = (V, E)$ as above be a semiframe. We show that in this case there exists a frame $H$ such that $\lambda\mu$ is a submetric of $\mathrm{d}^H$ for some $\lambda > 0$.

A complete bipartite graph with parts $A$ and $B$ is denoted by $(A;B)$. A maximal complete bipartite subgraph $K=(A;B)$ in $G$ is called a *bi-clique* if $|A|,|B|\ge 2$. First of all we observe the following.

**Statement 4.1.** *Let $K=(A;B)$ and $K'=(A';B')$ be two different bi-cliques such that $K\cap K'$ is nonempty. Then $K\cap K'$ is connected and contains at most one edge. In particular, every 4-cycle of $G$ is contained in exactly one bi-clique.*

*Proof.* Let for definiteness $A\cap A'\ne\emptyset$. Suppose that $A\cap A'$ contains two different nodes $x,y$. Since $K$ and $K'$ are different, w.l.o.g. we may assume that there are $u\in A$ and $v\in B'$ which are not adjacent in $G$. Choose two different nodes $z,z'$ in $B$ (existing because $|B|\ge 2$). Then the subgraph induced by $\{x,y,u,v,z,z'\}$ is $K_{3,3}^-$, contradicting the fact that $G$ is a semiframe. Thus, $|A\cap A'|=1$. Similarly, $|B\cap B'|\le 1$, and the result follows. •

The core of our construction of the desired frame $H$ involves orbit splitting operations that we now describe. Recall that an orbit of $G$ is a maximal set of dependent edges in it; since $G$ has no isometric $k$-cycle with $k>4$, the dependency relation involves only 4-cycles. The simplest case of an orbit is a bridge of $G$ (i.e., an edge whose removal makes $G$ disconnected). Each non-bridge edge belongs to a 4-cycle and, therefore, to a unique bi-clique (by Statement 4.1).

Consider an orbit $Q$. Let $\mathcal{K}(Q)$ be the set of bi-cliques whose edge sets meet $Q$. It is easy to see that all edges of a bi-clique $K=(A;B)\in\mathcal{K}(Q)$ with $|A|+|B|\ge 5$ are dependent; so they are entirely contained in $Q$. On the other hand, if $|A|=|B|=2$, it is possible that $Q$ includes only one pair of opposite edges of the 4-cycle $K$; in this case the bi-clique $K$ is called *simple*. The *orbit splitting operation* for $Q$ transforms $G$ into $G'=(V',E')$ as follows.

(4.1) (i) Split each edge $e=xy\in Q$ into two edges $xz_e$ and $yz_e$ in series.

(ii) If $K\in\mathcal{K}(Q)$ is simple and $K\cap Q=\{e,e'\}$, connect $z_e$ and $z_{e'}$ by an edge (see Fig. 6a).

(iii) If $K=(A;B)\in\mathcal{K}(Q)$ is non-simple, add a new node $v_K$ and connect it by an edge to $z_e$ for each edge $e$ of $K$ (see Fig. 6b where $|A|=2$ and $|B|=3$).

One can see that each simple bi-clique in $\mathcal{K}(Q)$ generates two bi-cliques (4-cycles) and each non-simple bi-clique $(A;B)\in\mathcal{K}(Q)$ generates $|A|+|B|$ bi-cliques in $G'$. Moreover, these new bi-cliques together with the bi-cliques of $G$ not in $\mathcal{K}(Q)$ constitute the full list of bi-cliques of $G'$. From this fact one can deduce that $G'$ is bipartite.

We call edges $xz_e$ and $yz_e$ in (i) of (4.1) *split-edges*, edges $z_ez_{e'}$ in (ii) *bridge-edges*, and edges $z_ev_K$ in (iii) *star-edges*. Let us say that an orbit $Q$ of $G$ is *orientable* if there exists a *feasible* orientation of the edges of this orbit, i.e., each two opposite edges of any 4-cycle of $G$ either are not oriented or have different orientations along this cycle (this matches the definition of the orientability in the Introduction). Note that $Q$ may be orientable while the whole $G$ is not. The graph $G'$ has important properties exhibited in the following three lemmas.

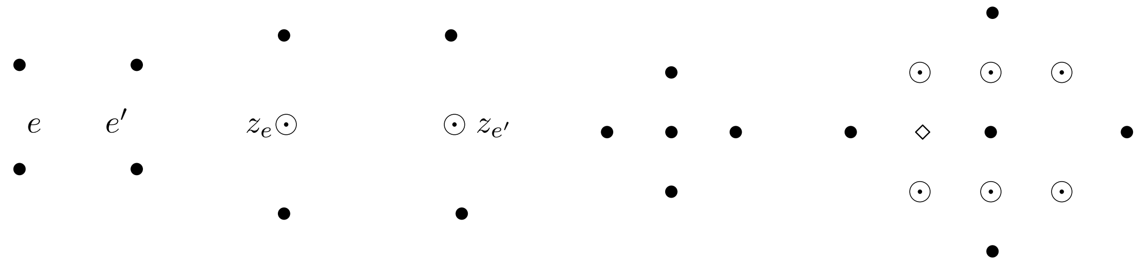


Fig. 6 (a) (b)

**Lemma 4.2.** *$G'$ is a semiframe.*

**Lemma 4.3.** *Let $Q_1, \dots, Q_r$ be the orbits of $G$, and let $G' = (V', E')$ be obtained from $G$ by the orbit splitting operation applied to $Q_1$. Then $G'$ has $r$ or $r+1$ orbits. Moreover,*

*(i) for $i = 2, \dots, r$, $Q_i$ induces an orbit $Q'_i$ in $G'$ formed by the edges of $Q_i$ and all bridge-edges $z_e z_{e'}$ such that the 4-cycle in $G$ containing $e, e'$ has the other two edges in $Q_i$; also $Q'_i$ is orientable if and only if $Q_i$ is so;*

*(ii) if $Q_1$ is non-orientable, then $Q_1$ induces one orbit $Q'_1$ in $G'$, which is orientable and formed by all the split- and star-edges;*

*(iii) if $Q_1$ is orientable, then $Q_1$ induces two orbits $Q'_1$ and $Q''_1$ in $G'$, and the set $Q'_1 \cup Q''_1$ consists of all the split- and star-edges; also both $Q'_1$ and $Q''_1$ are orientable, and for each $e = xy \in Q_1$, one of the edges $xz_e, yz_e$ belongs to $Q'_1$ while the other to $Q''_1$.*

**Lemma 4.4.** *Let $\rho$ be the length function on the edges of $G$ defined as follows (using notation from the previous lemma):*

*(4.2) (i) for $i = 2, \dots, r$, the $\rho$-length of each edge in $Q'_i$ is equal to the distance $m(e)$ of an edge $e \in Q_i$;*

*(ii) if $Q_1$ is non-orientable and $e \in Q_1$, then the $\rho$-length of each edge in $Q'_i$ is equal to $m(e)/2$;*

*(iii) if $Q_1$ is orientable and $e \in Q_1$, then fix an arbitrary number $0 < \alpha < m(e)$ and put $\rho(e') = \alpha$ for all $e' \in Q'_1$ and $\rho(e'') = m(e'') - \alpha$ for all $e'' \in Q''_1$.*

*Then the path metric $m^\rho = \mathrm{d}^{G',\rho}$ on $V'$ coincides with $\rho$ on $E'$ and is a tight extension of $\mu$.*

These lemmas will be proved later, and now we explain how they help us to find the required frame $H$.

First we apply the orbit splitting operation consecutively to each of the orbits $Q_1, \ldots, Q_r$ of $G$ (more precisely, to the images of $Q_i$'s in the current graphs). This results in a semiframe $\widetilde{G} = (\widetilde{V}, \widetilde{E})$ and a function $\widetilde{\rho}$ on $\widetilde{E}$ such that $\widetilde{m} = d^{\widetilde{G}, \widetilde{\rho}}$ is an extension of $\mu$ (by repeatedly applying Lemmas 4.2-4.4). One can see that $\widetilde{G}$ can also be directly constructed from $G$ as follows (cf. (4.1)).

(4.3) (i) Split *each* edge $e \in E$ into two edges $xz_e$ and $yz_e$ in series.

(ii) For *each* bi-clique $K$ of $G$, add a new node $v_K$ and edges $z_e v_K$ for all edges $e$ of $K$.

This enables us to show the following important property.

**Statement 4.5.** $\widetilde{G}$ *is a frame.*

*Proof.* We observe from (4.3) that each 4-cycle $\widetilde{C}$ of $\widetilde{G}$ is of the form $xz_e v_K z_{e'} x$, where $e$ and $e'$ are edges in a bi-clique $K$ of $G$ which are incident to a node $x$. Therefore, we can orient each split-edge $xz_e$ from $x$ to $z_e$ and each star-edge $z_e v_K$ from $z_e$ to $v_K$, obtaining a feasible orientation for all 4-cycles of $\widetilde{G}$. Thus, $\widetilde{G}$ is orientable, and now the fact that $\widetilde{G}$ is a semiframe (by Lemma 4.2) implies that $\widetilde{G}$ is a frame. •

**Remark 4.6.** In fact, in the above method we do not need to split *all* orbits of $G$ to transform it into a frame. Lemma 4.3 shows that it suffices to split only each non-orientable orbit. We will use this fact in Section 5.

Next, since $\mu$ can be considered up to proportionality, we may assume that all numbers $m(e)$, $e \in E$, are even integers (this is slightly stronger than the above assumption that $\mu$ is cyclically even). We also assume that for each orientable orbit of $G$ the number $\alpha$ figured in (iii) of Lemma 4.4 is chosen to be an integer. Then the function $\widetilde{\rho}$ (as well as metric $\widetilde{m}$) is integer-valued. We now repeatedly split each orbit of $\widetilde{G}$ so as to get a frame with unit lengths of all edges.

More precisely, starting from $\overline{G} = \widetilde{G}$ and $\overline{\rho} = \widetilde{\rho}$, choose an orbit $Q$ of the current graph $\overline{G}$ such that the current length $\overline{\rho}(e) =: \delta$ of an edge $e \in Q$ is still greater than one. Split $Q$ with an arbitrary integer $\alpha$ ($0 < \alpha < \delta$); this transforms $Q$ into two orbits $Q'$ and $Q''$ with length $\alpha$ of all edges in $Q'$ and length $\delta - \alpha$ of all edges in $Q''$ (taking into account that $Q$ is orientable, by Lemma 4.3). Choose an appropriate orbit in the new current graph and do similarly, and so on. Eventually, we obtain a frame $H = (W, U)$ with unit length of each edge (as before, this follows by induction, using Lemmas 4.2-4.4). The resulting metric on $W$ is just $d^H$ and this metric is an extension of $\mu$ (by Lemma 4.4). So $\mu$ is a submetric of $\mathrm{d}^H$, yielding (ii) in Theorem 1.2, as required.

It remains to prove Lemmas 4.2-4.4 (at the first glance, these lemmas look rather transparent; however, they will take some technical efforts to be carefully proved). Let $Q_1,\ldots,Q_r$ be the orbits of $G$, and let $G'=(V',E')$ be formed by the orbit splitting operation applied to $Q=Q_1$.

*Proof of Lemma 4.3.* (i) in this lemma is easily shown by use of Statement 4.1.

To see (iii), fix a feasible orientation of the edges of $Q$ and form $Q'$ and $Q''$ as follows. If $e=xy\in Q$ is oriented as $(x,y)$, then orient $xz_e$ as $(x,z_e)$ and include it in $Q'$, and orient $yz_e$ as $(z_e,y)$ and include it in $Q''$.

To assign membership for the star-edges, consider a non-simple bi-clique $K=(A;B)\in\mathcal{K}(Q)$, and choose a 4-cycle $C=v_0v_1v_2v_3v_0$ in $K$. Assume that the edges of $C$ are oriented as drawn in Fig. 1, and let $v_0,v_2\in A$. Then $|A|=2$ (otherwise the above orientation for $C$ cannot be extended to a feasible orientation in $K$), and for each $x\in B$, the edges $v_0x$ and $v_2x$ must be oriented as $(v_0,x)$ and $(x,v_2)$. For each $x\in B$ and $e=v_0x$, orient $z_ev_K$ as $(z_e,v_K)$ and include it in $Q'$, while for $e'=v_2x$, orient $z_{e'}v_K$ as $(v_K,z_{e'})$ and include it in $Q''$.

One can check that each of $Q',Q''$ is indeed an orbit of $G'$ and that the orientation we constructed is feasible for both $Q'$ and $Q''$.

To see (ii), orient each split-edge $xz_e$ as $(x,z_e)$ and each star-edge $z_ev_K$ as $(z_e,v_K)$. This gives a feasible orientation of the split- and star-edges (compare with the orientation in the proof of Statement 4.5). Next, since $Q$ is non-orientable, for any edge $e\in Q$, there exists a sequence (*orientation-reversing dual cycle*) $D=(e^0,F^1,e^1,\ldots,F^q,e^q)$ consisting of edges $e^i=x^iy^i$ and 4-cycles $F^i$ of the form $x^{i-1}y^{i-1}y^ix^ix^{i-1}$ in $G$ such that $e=e^0=e^q$, $x^0=y^q$ and $y^0=x^q$. One can see that the edges $x^0z_{e^0},x^1z_{e^1},\ldots,x^qz_{e^q}$ of $G'$ are dependent, and now the fact that $x^qz_{e^q}=y^0z_{e^0}$ (as $x^q=y^0$) shows that all split-edges generated by the edges of $D$ are dependent. This easily implies that all split- and star-edges in $G'$ are dependent, and hence, they constitute one orbit of $G'$. •

It is convenient to prove the other two lemmas together because their proofs involve some common arguments.

*Proof of Lemmas 4.2 and 4.4.* It falls into several claims. Let $m'=m^\rho$. We call nodes of $G'$ of the form $z_e$ and $v_K$ *split-* and *star-nodes*, respectively.

**Claim 1.** *$m'$ is an extension of $m$ (and therefore, an extension of $\mu$).*

*Proof.* Consider a $V$-path $P=x_0\ldots x_k$ in $G'$. We show that $\rho(P)\geq m(x_0x_k)$. Apply induction on $|P|$. One may assume that $P$ is simple and $|P|\geq 3$ (for if $k=1$, then $e = x_0x_1 \in E$ and $\rho(P) = m(e)$, and if $k = 2$ and $P$ is not a path of $G$, then $e = x_0x_2 \in E$ and $x_1 = z_e$, whence $\rho(P) = m(e)$, by (4.2)(ii),(iii)). Also one may assume that no intermediate node of $P$ occurs in $V$; otherwise partition $P$ into two $V$-paths and apply induction.

We show that $P$ can be transformed into an $x_0$–$x_k$ path $P'$ in $G'$ such that either $|P'| < |P|$ and $\rho(P') < \rho(P)$, or $|P'| = |P|$, $\rho(P') = \rho(P)$ and $P'$ has an intermediate node in $V$. In both cases the result follows by induction.

Obviously, $x_1$ is a split-node $z_e$, where $e = x_0y \in E$. If $x_2$ is also a split-node $z_{e'}$, then $z_ez_{e'}$ is a bridge-edge. Hence, there is a 4-cycle $C = x_0yy'x'x_0$ of $G$ with $e' = x'y'$. Then $x_0x_1x_2x'x_0$ is a 4-cycle of $G'$, and we have $\rho(x_0x_1) = \rho(x_2x')$ and $\rho(x_0x') = \rho(x_1x_2)$. So the path $P' = x_0x'x_2 \dots x_k$ is as required (it contains $x' \in V$ as an intermediate node).

Now suppose that $x_2$ is a star-node $v_K$. Then $x_3$ is a split-node $z_{e'}$, and both $e, e'$ belong to a 4-cycle $C = x_0yy'x'x_0$ in $K$. Three cases are possible.

(i) $e' = x_0x'$. Then $x_0x_1x_2x_3x_0$ is a 4-cycle in $G'$. Since $\rho(x_0x_3) = \rho(x_1x_2)$ (by (4.2)), the path $P' = x_0x_3x_4 \dots x_k$ satisfies $|P'| < |P|$ and $\rho(P') < \rho(P)$.

(ii) $e' = yy'$. Then $x_1x_2x_3yx_1$ is a 4-cycle in $G'$, whence $\rho(x_1x_2) = \rho(yx_3)$ and $\rho(x_1y) = \rho(x_2x_3)$. Hence, the path $P' = x_0x_1yx_3 \dots x_k$ in $G'$ satisfies $|P'| = |P|$ and $\rho(P') = \rho(P)$ and contains $y \in V$ as an intermediate node.

(iii) $e' = x'y'$. Let $e'' = x_0x'$ and $z = z_{e''}$. Then both $x_0x_1x_2zx_0$ and $zx_2x_3x'z$ are 4-cycles in $G'$, whence $\rho(x_0x_1) = \rho(zx_2) = \rho(x'x_3)$, $\rho(x_1x_2) = \rho(x_0z)$ and $\rho(x_2x_3) = \rho(zx')$. Therefore, the path $P' = x_0zx'x_3 \dots x_k$ is as required.

Thus, $m'|_V \ge m$. To see equality, let $P = x_0 \dots x_k$ be an $m$-shortest path in $G$, and let $\gamma(P)$ be the path in $G'$ obtained from $P$ by replacing each edge $e = x_{i-1}x_i$ occurring in $Q$ by $x_{i-1}z_ex_i$. For such an edge $e$, we have $\rho(x_{i-1}z_e) + \rho(z_ex_i) = m(e)$. This implies $m(P) = \rho(\gamma(P))$. •

A path $P'$ in $G'$ is called *regular* if $P' = \gamma(P)$ for some path $P$ in $G$, where $\gamma$ is as above. Arguing as in the prove of Claim 1, we have:

(4.4) for each $m$-shortest path $P$ in $G$, the path $\gamma(P)$ is $\rho$-shortest, and $\rho(\gamma(P)) = m(P)$;

(4.5) for any $s$–$t$ path $L'$ in $G'$ with $s, t \in V$, there is a regular $s$–$t$ path $L$ such that $|L| \le |L'|$.

Two graphs will play a special role later on. The *grid* $\Gamma_{p,q}$, or $p \times q$ *grid*, is the graph whose nodes correspond to the vectors $(i, j)$ for $i = 0, 1, \dots, p$ and $j = 0, 1, \dots, q$ and edges correspond to the pairs $\{(i, j), (i', j')\}$ with $|i - i'| + |j - j'| = 1$. A part

$\Gamma$ of $\Gamma_{p,q}$ bounded by two shortest paths from $s = (0,0)$ to $t = (p,q)$ is called a *net* from $s$ to $t$, or an *$s$–$t$ net*. That is, $\Gamma$ is a subgraph of $\Gamma_{p,q}$ induced by the nodes $(i,j)$ satisfying $a_j \le i \le b_j$ for two sequences $0 = a_0 \le a_1 \le \ldots \le a_q \le p$ and $0 \le b_0 \le b_1 \le \ldots \le b_q = p$ with $a_j \le b_j$, $j = 0, \ldots, q$. Figure 7b illustrates a net $\Gamma$ for $p = 4$ and $q = 3$. The *rightmost* (resp. *leftmost*) path from $s$ to $t$ in $\Gamma$ is denoted by $R^\Gamma$ (resp. $L^\Gamma$). Sometimes a node with coordinates $(i,j)$ in $\Gamma$ is denoted by $(i,j)_\Gamma$.

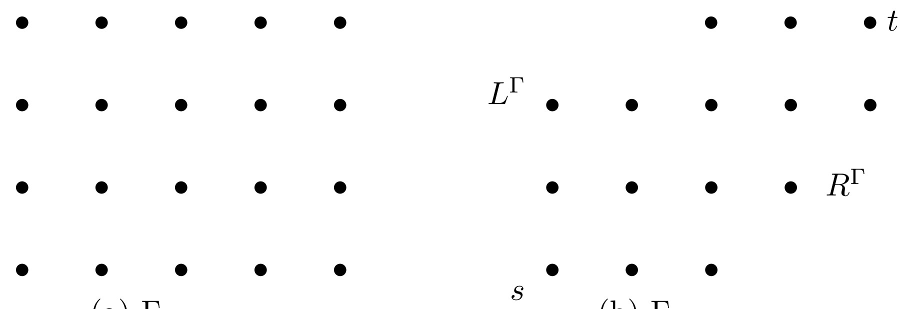


Fig. 7 (a) $\Gamma_{4,3}$ (b) $\Gamma$

We will use as an important tool the property that any two shortest paths with the same ends in $G$ can be linked by an isometric net. More precisely,

(4.6) (i) for any $s, t \in V$ and shortest $s$–$t$ paths $P$ and $P'$ in $G$, there exists an $s$–$t$ net $\Gamma$ in $G$ such that $R^\Gamma = P$ and $L^\Gamma = P'$;

(ii) any 2-connected net in $G$ is isometric.

This property was proved in [10] concerning the frames; however, the proof remains valid for the semiframes as it does not really uses the orientability of the hereditary modular graph in question but only absence of induced subgraphs $K^-_{3,3}$ in it. We outline the proof for completeness of our description.

*Sketch of the proof of (4.6).* We show (i) by induction on $|P|$. Let $P = sx_1 \ldots x_k t$ and $P' = sy_1 \ldots y_k t$. Case $|P| \le 2$ is obvious, so assume $|P| \ge 3$. Also one may assume that $x_i \ne y_j$ for $i, j = 1, \ldots, k$ (otherwise the result easily follows by induction). Since $G$ has no isometric $n$-cycle with $n > 4$, there are $i, j$ such that $\mathrm{d}^G(x_i y_j) < i+j, 2k+2-i-j$; one may assume that $i + j \le 2k + 2 - i - j$, that $i \ge j$, and that $i + j$ is minimum.

Choose a shortest path $y_j z_1 \ldots z_q x_i$ in $G$. The above assumptions and the bipartiteness of $G$ imply that the paths $L = y_{j-1} \ldots y_1 s x_1 \ldots x_i$, $L' = y_{j-1} y_j z_1 \ldots z_q x_i$ and $L'' = y_j \ldots y_1 s x_1 \ldots x_{i-1}$ are shortest (letting $x_0 = y_0 = s$). By induction there is a $y_{j-1}$–$x_i$ net $\Gamma'$ with $R^{\Gamma'} = L$ and $L^{\Gamma'} = L'$. Clearly $L''$ can be shortest only if $\Gamma$ is the $(i + j - 2) \times 1$ grid in which $y_j, x_{i-1}, x_i, s$ have the coordinates $(0,1), (0, i+j-2), (1, i+j-2), (j-1, 0)$, respectively (in view of $i \ge j$). Then $D = s z_{j-1} \ldots z_q x_i \ldots x_k t$ and $D' = s z_{j-1} \ldots z_1 y_j \ldots y_k t$ are shortest paths in $G$. By induction there is a $z_{j-1}$–$t$ net $\Gamma''$ with $R^{\Gamma''}$ and $L^{\Gamma''}$ to be the parts of $D$ and $D'$ from

$z_{j-1}$ to $t$, respectively.

We assert that $j=1$ (and $s=(0,0)_{\Gamma'}$). Indeed, if $j>1$, then $G$ contains the 6-cycle $sx_1z_jvz_{j-2}y_1s$, where $v=(1,1)_{\Gamma''}$, taking into account that $z_j=(1,0)_{\Gamma''}$ and $z_{j-2}=(0,1)_{\Gamma''}$ (letting $z_0=y_j$ and $z_{q+1}=x_i$). Since $C$ is not isometric, some pair of opposite nodes of $C$ is connected by an edge. Then the facts that $z_{j-1}$ is adjacent to $z_j, z_{j-2}, s$, while $s$ and $v$ are not adjacent (as $z_{j-1}$ and $v$ occur in a shortest $s$–$t$ path in this order) imply that $G$ contains an induced $K_{3,3}^-$; a contradiction. Thus, $j=1$. Next, for $r=0,\ldots,q+1$, we have $z_r=(r,0)_{\Gamma''}$ (otherwise for some $1\le r\le q$, the nodes $z_{r-1}, z_r, z_{r+1}$ are $(r-1,0), (r,0), (r,1)$ in $\Gamma''$, respectively, and we can reveal an induced $K_{3,3}^-$ in $G$ by considering the 6-cycle $x_rx_{r+1}z_{r+1}wz_{r-1}x_{r-1}x_r$, where $w=(r-1,1)_{\Gamma''}$). Now the union of $\Gamma'$ and $\Gamma''$ gives the desired $s$–$t$ net $\Gamma$.

To see (ii), suppose that some 2-connected net $\Gamma$ is not isometric and choose $x=(p,q)_\Gamma$ and $y=(p',q')_\Gamma$ with $\Delta := |p-p'|+|q-q'|$ minimum provided that $\Delta > \mathrm{d}^G(xy)$. Let $L=x_0\ldots x_\Delta$ be an $x$–$y$ path of length $\Delta$ in $\Gamma$. Since $\Gamma$ is 2-connected and $\Delta\ge 3$, $L$ can be chosen so that some of $p_0=p_2$, $q_0=q_2$, $p_\Delta=p_{\Delta-2}$ $q_\Delta=q_{\Delta-2}$ holds. Let for definiteness $p_0=p_2$ and $q_2=q_0+2$. Since $\Gamma$ is 2-connected, it contains nodes $v_i=(\overline{p},q_i)$, $i=0,1,2$, where $\overline{p}\in\{p_0-1,p_0+1\}$. Let $P$ be the concatenation of $x_1x$ and a shortest $x$–$y$ path in $G$. By the minimality of $x,y$, the paths $P$ and $L'=x_1\ldots x_\Delta$ are shortest, so there is an $x_1$–$y$ net $\Gamma'$ with $R^{\Gamma'}=P$ and $L^{\Gamma'}=L'$, by (i). Considering the 6-cycle $xv_0v_1v_2x_2ux$, where $u=(1,1)_{\Gamma'}$, we can reveal an induced $K_{3,3}^-$; a contradiction. •

Property (4.6) enables us to show the following.

**Claim 2.** *For $s,t\in V$, let $P$ be a shortest $s$–$t$ path and $P'$ be an $s$–$t$ path in $G$. Then $|P\cap Q|\le|P'\cap Q|$ and $|P-Q|\le|P'-Q|$.*

*Proof.* First suppose that $P'$ is also shortest. Take a net $\Gamma$ as in (4.6), and let $s=(0,0)_\Gamma$ and $t=(p,q)_\Gamma$. Observe that any two edges $e=(i,j)(i+1,j)$ and $e'=(i,j')(i+1,j')$ are dependent; so $e\in Q$ if and only if $e'\in Q$. Similarly, any two edges $(i,j)(i,j+1)$ and $(i',j)(i',j+1)$ are dependent. This implies $|R^\Gamma\cap Q|=|L^\Gamma\cap Q|$.

Next suppose that $|P'|\ge|P|$; let $P=sx_1\ldots x_kt$ and $P'=sy_1\ldots y_qt$. We use induction on $|P'|$. One may assume that no intermediate node of $P'$ is in $P$. Take a median $v$ for $\{s,x_k,y_q\}$. In light of the previous case, we can consider as $P$ any shortest $s$–$t$ path. So one may assume that $P$ contains $v$ (as $v$ is in a shortest $s$–$x_k$ path); let $v=x_{k'}$. By (2.2), the path $x_kty_q$ is shortest, and a shortest $x_k$–$y_q$ path $D$ containing $v$ satisfies $|D|=2$. If $k'=k$, then the path $L=sx_1\ldots x_kty_q$ is shortest, and now the required inequalities for $P$ and $P'$ immediately follow by induction from those for $L$ and $P''=sy_1\ldots y_q$.

Now let $k' \neq k$. Then $D = x_k v y_q$, whence $k' = k-1$. Applying induction to the shortest path $R = sx_1 \dots x_{k-1} y_q$ and the path $P''$ as above, we have $|R \cap Q| \leq |P'' \cap Q|$ and $|R - Q| \leq |P'' - Q|$. Also $x_{k-1}x_k \in Q$ if and only if $y_q t \in Q$, and similarly for $x_{k-1}y_q$ and $x_k t$ (regarding the 4-cycle $x_{k-1}x_k t y_q x_{k-1}$). Therefore, $|P' \cap Q| - |P \cap Q| = |P'' \cap Q| - |R \cap Q|$, and the result follows. •

This claim together with (4.4), (4.5) and Statement 2.2 shows that

(4.7) a path $P$ in $G$ is shortest if and only if $\gamma(P)$ is $G'$-shortest and if and only if $\gamma(P)$ is $\rho$-shortest.

Let $d$ and $d'$ stand for the metrics $\mathrm{d}^G$ and $\mathrm{d}^{G'}$, respectively. In order to prove the remaining statements in Lemmas 4.2 and 4.4, we need the following corollary from (4.6) (which occurred in [10] for frames):

(4.8) if $a, c, b_1, b_2 \in V$ are different nodes such that $ab_i \in E$ and $d(b_i c) < d(ac)$ for $i = 1, 2$, then there is a unique node $a' \in V$ such that $a'b_1, a'b_2 \in E$ and $d(ac) = d(a'c) + 2$.

Indeed, take an $a$–$c$ net $\Gamma$ with $R^\Gamma$ containing $b_1$ and $L^\Gamma$ containing $b_2$. Then $a' = (1,1)_\Gamma$ is adjacent to $b_1$ and $b_2$ and satisfies $d(ac) + d(a'c) + 2$. Suppose that there is another node $a''$ with a similar property. Then $\Gamma$ can be chosen so that $R^\Gamma$ passes $b_1, a'$ and $L^\Gamma$ passes $b_2, a''$. Since $a' \neq a''$, w.l.o.g. one may assume that $a'' = (0,2)_\Gamma$ (and $b_1 = (1,0)_\Gamma$ and $b_2 = (0,1)_\Gamma$). By (4.6)(ii), the subnet of $\Gamma$ induced by the points $(i,j)$ for $i = 0,1$ and $j = 0,1,2$ is isometric; therefore, $d(a''b_1) = 3$. So $a''$ cannot be adjacent to $b_1$; a contradiction.

The further proof will rely on the following key claim. For $x \in V'$, let $N(x)$ denote the set of nodes in $G$ closest to $x$, i.e., $N(x)$ is $\{x\}$ if $x \in V$, $\{u, v\}$ if $x$ is a split-node $z_{uv}$, and the node set of the bi-clique $K$ if $x$ is a star-node $v_K$. If $(x_0, x_1, \dots, x_k)$ is a sequence of nodes which occur in this order in a shortest path in $G$ (resp. $G'$), we call this sequence $G$-shortest (resp. $G'$-shortest).

**Claim 3.** *Let $x, y \in V'$. Then there exist $s \in N(x)$ and $t \in N(y)$ such that $d'(sx) + d'(xy) + d'(yt) = d'(st)$, i.e., $(s, x, y, t)$ is $G'$-shortest.*

*Proof.* One may assume that $x \neq y$; otherwise the result is trivial. For $z \in V'$, let $\varphi(z)$ denote $d'(zv)$, where $v \in N(z)$, i.e., $\varphi(z)$ is 0 if $z \in V$, 1 if $z$ is a split-node, and 2 if $z$ is a star-node. First of all we show that

(4.9) if $z \in V'$ and $u \in V$, and if $v$ is an element of $N(z)$ with $d(uv)$ maximum, then $(u, v', z, v)$ is $G'$-shortest for some $v' \in N(z)$, or, equivalently, $d'(uv) = d'(uv') +$

$2\varphi(z)$.

This is obvious if $z \in V$. Suppose that $z$ is a split-node $z_{vv'}$. Since $G$ is bipartite, $d(uv) \neq d(uv')$; therefore, $d(uv) > d(uv')$ (by the maximality of $v$), and $(u, v', v)$ is $G$-shortest. This implies that $(u, v', z, v)$ is $G'$-shortest, by (4.7). Now suppose that $z$ is a star-node $v_K$ for $K = (A; B)$. Let for definiteness $v \in A$. Choose two different nodes $p_1, p_2 \in B$. Then $d(up_i) < d(uv)$. By (4.8) for $v, u, p_1, p_2$, there is $v' \in V$ adjacent to both $p_1, p_2$ and satisfying $d(uv') = d(uv) - 2$. Then $v \in N(z)$ (by Statement 4.1) and $(u, v', v)$ is $G$-shortest, whence $(u, v', z, v)$ is $G'$-shortest.

Note that (4.9) is equivalent to $d(uv) = d(uv') + \varphi(z)$ (in view of (4.7)). We choose the desired $s \in N(x)$ and $t \in N(y)$ so that the distance $d(st)$ (or $d'(st)$) is maximum. By (4.9), there are $s' \in N(x)$ and $t' \in N(y)$ such that each of $(s, x, s', t)$ and $(s, t', y, t)$ is $G'$-shortest. If $(s, s', t', t)$ is shortest, we are done. In particular, this happens if at least one of $x, y$ is in $V$. So assume that $d(st) < d(s't') + \varphi(x) + \varphi(y)$ and consider possible cases.

(i) Suppose that both $x, y$ are split-nodes. Then $x = z_{ss'}$ and $y = z_{tt'}$. Also $\varphi(x) = \varphi(y) = 1$, and $d(st) - d(s't')$ is an even integer $\leq 2$. So we need to consider only the case when $d(s't') = d(st) =: \alpha$. Take an $s$–$t$ net $\Gamma$ in $G$ with $R^\Gamma$ containing $s'$ and $L^\Gamma$ containing $t'$. Observe that $d(s't') = d(st)$ is possible only if $\Gamma$ is a $1 \times (\alpha - 1)$ grid in which $s' = (1, 0)_\Gamma$ and $t' = (0, \alpha - 1)_\Gamma$. Then the edges of $\Gamma$ of the form $(0, j)(1, j)$ belong to the orbit $Q$. The split-nodes induced by these edges together with the star-nodes induced by the non-simple bi-cliques including 4-cycles from $\Gamma$ generate in a natural way an $x$–$y$ path $P'$. Moreover, the concatenation of $sx$, $P'$ and $yt$ gives an $s$–$t$ path $P$ which, obviously, has the same length as that of $\gamma(R^\Gamma)$. Hence, $(s, x, y, t)$ is $G'$-shortest.

(ii) Next suppose that both $x, y$ are star-nodes, $x = v_{K=(A;B)}$ and $y = v_{K'=(A';B')}$. Assuming that $s \in A$ and $t \in A'$, we have $s' \in A$ and $t' \in A'$. Let $\alpha = d(st)$. Since $\varphi(x) + \varphi(y) = 4$, we need to consider two cases, namely, $d(s't') = \alpha$ and $\alpha - 2$.

(a) Let $d(s't') = \alpha$. Consider an $s$–$t$ net $\Gamma$ such that $R^\Gamma$ contains $s'$ and $L^\Gamma$ contains $t'$. Observe that $d(s't') = d(st)$ is possible only if $\Gamma$ is a $2 \times (\alpha - 2)$ grid in which $s' = (2, 0)_\Gamma$ and $t' = (0, \alpha - 2)_\Gamma$. Note that $\alpha \geq 3$ (otherwise $K = K'$ and $x = y$). Consider the subnet $\Gamma'$ of $\Gamma$ induced by the points $(i, j)$ for $i = 0, 1, 2$ and $j = 0, 1$, and take a node $q \in B$ different from $(1, 0)_\Gamma$ ($q$ exists since $|B| \geq 2$). Since $\Gamma'$ is isometric (by (4.6)(ii)) and $G$ is bipartite, $q$ is different from all nodes of $\Gamma'$. Let $H$ be the subgraph of $G$ induced by the nodes of $\Gamma'$ and $q$. If $q$ is adjacent to $(1, 1)_\Gamma$, then $H$ contains $K_{3,3}^-$, and if not, then $H$ contains an isometric 6-cycle. Thus, the given case is impossible.

(b) Let $d(s't') = \alpha - 2$. Choose $p \in B$ and $q \in B'$ with $d(pq)$ maximum; clearly

$d(pq) \in \{\alpha, \alpha-2\}$. If $d(pq)=\alpha$, take an $s$–$t$ net $\Gamma$ in $G$ with $R^\Gamma$ containing $p, s'$ and $L^\Gamma$ containing $q, t'$. Since $d(st)=d(pq)$, $\Gamma$ is a $1\times(\alpha-1)$ grid in which $p, s', t', q$ have the coordinates $(1,0), (1,1), (0,\alpha-2)$ and $(0,\alpha-1)$, respectively. This implies that $(s,x,y,t)$ is $G'$-shortest (by the argument similar to that in (i)). Now let $d(pq)=\alpha-2$. Then $(s,p,q,t)$ is $G$-shortest. Choose $p' \in B-\{p\}$ and $q' \in B'-\{q\}$. We have $d(p'q)=d(pq)<d(sq)$ (by the choice of $p,q$); hence, there is $s'' \in A$ such that $d(s''q)=\alpha-3$ (by (4.8)). Similarly, there is $t'' \in A'$ such that $d(t''p)=\alpha-3$. We can consider $s'', t''$ instead of $s', t'$; so we may assume that $d(s''t'')=\alpha-2$. Then a $p$–$q$ net $\Gamma$ in $G$ with $R^\Gamma$ containing $s''$ and $L^\Gamma$ containing $t''$ is a $1\times(\alpha-3)$ grid. One can see that none of $s,t,p',q'$ occurs in $\Gamma$. Therefore, adding to $\Gamma$ these nodes and the edges $sp, sp', p's'', tq, tq', q't''$ makes a $1\times(\alpha-1)$ grid $\Gamma'$. Since $\Gamma'$ is isometric (by (4.6)(ii)), $d(p'q')=\alpha$, contrary to the choice of $p,q$.

The remaining case when one of $x,y$ is a split-node and the other is a star-node combines arguments from (i) and (ii) and is left to the reader as an exercise. •

This claim shows that any two nodes $x,y \in V'$ are contained in a $G'$-shortest $s$–$t$ path $L$ for some $s \in N(x)$ and $t \in N(y)$. Moreover, starting from an arbitrary $L$ with such a property, one can transform it, step by step, into a regular $G'$-shortest $s$–$t$ path $L'$ (not necessarily including $x$ and $y$) by performing elementary transformations described in the proof of Claim 1. We have seen that each elementary transformation does not change the $\rho$-length of a path (when its $G'$-length preserves). This together with the fact that the resulting path $L'$ is $\rho$-shortest (by (4.7)) shows that the initial path $L$ is $\rho$-shortest as well. Thus, $m^\rho(sx)+m^\rho(xy)+m^\rho(yt)=m^\rho(st)=m(st)$ (in view of Claim 1), i.e., $m^\rho$ is a tight extension of $m$ and, therefore, $m^\rho$ is a tight extension of $\mu$. Also, considering $L$ for the case when $xy$ is an edge of $G'$, we conclude that $\rho(xy)=m^\rho(xy)$. So Lemma 4.4 is proven.

We now use Claim 3 to show that $G'$ is a semiframe.

**Claim 4.** *$G'$ is modular.*

*Proof.* By (iii) in Theorem 1.8, it suffices to prove the existence of a median for a triple $\{u_1,u_2,y\}$ in $V'$ such that $d'(u_1u_2)=2$ and $d'(u_1y)=d'(u_2y)\geq 2$. Let $x \in V'$ be adjacent to both $u_1,u_2$. If $d'(xy)=d'(u_1y)-1$, then $x$ is a median for $u_1,u_2,y$. So assume that $d'(xy)=d'(u_iy)+1$, i.e., $(x,u_i,y)$ is $G'$-shortest. We will use many times the following corollary from (4.7) and the bipartiteness of $G$:

(4.10) if $(a,b,c)$ is $G'$-shortest, $a,c \in V$, and $b$ is a split-node $z_{aa'}$, then $d(ac)>d(a'c)$ and, therefore, $(a,a',c)$ is $G$-shortest.

Indeed, if $(a,a',c)$ is not $G$-shortest, then $(a',a,c)$ is $G$-shortest (as $G$ is bipartite and $aa' \in E$). So $a$ occurs in a shortest $a'$–$c$ path $P$ in $G$. By (4.7), $\gamma(P)$ is $G'$-shortest. Hence, $(a', b, a, c)$ is $G'$-shortest, contradicting the fact that $(a, b, c)$ is $G'$-shortest.

Choose $s, t$ as in Claim 3 for our $x, y$. Then $(s, x, u_i, y, t)$ is $G'$-shortest for $i = 1, 2$. Let $L_i$ be a $G'$-shortest $s$–$t$ path containing $x, u_i, y$. We examine possible cases for $x$.

(i) Suppose that $x$ is a star-node $v_K$ for $K = (A; B)$. Then $u_1 = z_e$ and $u_2 = z_{e'}$ for some edges $e = p_1q_1$ and $e' = p_2q_2$ in $K$. Let for definiteness $s, q_1, q_2 \in A$ (and $p_1, p_2 \in B$). Since $(s, x, u_i)$ is shortest, $d'(su_i) > 1$, whence $q_i \neq s$. Moreover, the $s$–$t$ path $L'_i$ obtained from $L_i$ by replacing its part from $s$ to $u_i$ by $sz_{sp_i}p_iu_i$ is again $G'$-shortest. So $(p_i, u_i, t)$ is $G'$-shortest, whence $(p_i, q_i, t)$ is $G$-shortest (by (4.10)). This implies that

(4.11) $$\text{for } i = 1, 2, \quad (s, p_i, q_i, t) \text{ is } G\text{-shortest.}$$

By (4.8), there is a unique node in $G$ adjacent to both $p_1, p_2$ and closer to $t$ than $p_i$. Hence, $q_1 = q_2 =: q$ (since any $p_i$ and $q_j$ are adjacent). If $y \in V$ (i.e., $y = t$), then $q$ is the desired median for $u_1, u_2, y$ (in view of (4.7) and (4.11)). We show that $q$ is a median in two remaining cases for $y$ as well.

Let $y$ be a split-node $z_{tt'}$. Since $s, p_i, y, t$ occur in this order in $L'_i$, $(s, p_i, t', t)$ is $G$-shortest (by (4.10) for $t, y, p_i$). By (4.8), there is a node $q' \in V$ adjacent to $p_1, p_2$ and such that $(p_i, q', t')$ is $G$-shortest. We assert that $q' = q$. Indeed, if $q' \neq q$, we can apply (4.8) to $p_1, t, q, q'$ (because of $d(qt) = d(q't) = d(p_1t) - 1$) to obtain a node $q''$ adjacent to $q, q'$ and satisfying $d(q''t) < d(qt)$. One can see that the subgraph of $G$ induced by $\{s, p_1, p_2, q, q', q''\}$ is $K^-_{3,3}$; a contradiction. Thus, $q = q'$, and now the fact that $(p_i, q', t', t)$ is $G$-shortest implies that $(u_i, q, y)$ is $G'$-shortest, whence $q$ is a median for $\{u_1, u_2, y\}$.

Now let $y$ be a star-node $v_{K'}$ for $K' = (A'; B')$. We may assume that $L'_i$ ends with $z_{t'h}v_{K'}z_{ht}t$. Then the path $z_{t'h}hz_{ht}t$ is also $G'$-shortest, whence $d'(p_it') < d'(p_it)$. This implies that $(p_i, t', t)$ is $G$-shortest, and we now proceed as in the previous case.

(ii) Next suppose that $x$ is a split-node $z_{ss'}$. Consider five possible cases, depending on the types of $u_1, u_2$ (symmetric cases are omitted).

(a) Let $u_1$ be a split-node $z_{pq}$ and $u_2 \in V$; then $u_2 = s'$. One may assume that $sp \in E$ (then $u_2q \in E$). Since the path obtained from $L_1$ by replacing the part $sxu_1$ by $spu_1$ is $G'$-shortest, $(p, q, t)$ is $G$-shortest (by (4.10) for $p, u_1, t$). Also $(u_2, q, t)$ is $G$-shortest. So $q$ is a median for $p, u_2, t$ in $G$, whence $q$ is a median for $u_1, u_2, t$ in $G'$. Now arguing as in (i), we conclude that $q$ is a median for $(u_1, u_2, y)$ too.

(b) Let $u_i$ be a split-node $z_{p_iq_i}$ for $i = 1, 2$ (i.e., $xu_i$ is a bridge-edge), and assume that both $p_1, p_2$ are adjacent to $s$. Then $(s, p_i, u_i, t)$ is $G'$-shortest, whence $(s, p_i, q_i, t)$ is $G$-shortest (by (4.10)). Take $q \in V$ adjacent to both $p_1, p_2$ and giving $d(qt) < d(p_it)$.

Note that all nodes in $Z = \{s, s', p_1, q_1, p_2, q_2\}$ are different (otherwise they belong to the same bi-clique of $G$, whence $x$ and $u_1$ cannot be adjacent). Also $q \neq q_1, q_2$ (otherwise the induced subgraph on $Z$ is $K_{3,3}^-$). Now considering the 6-cycle $C = qp_1q_1s'q_2p_2q$, one can see that either $C$ is isometric (when $qs' \notin E$) or $G$ contains an induced $K_{3,3}^-$ (otherwise). Thus, this case is impossible.

(c) Let $u_1$ be a split-node $z_{pq}$ and $u_2$ a star-node $v_K$. Assuming that $sp \in E$ and $sp'$ is an edge in $K$ with $p' \neq s'$, and considering corresponding $G'$-shortest $s$–$t$ paths, we observe that $d(pt), d(p't) < d(st)$. Now one shows that this case is impossible by arguing as in (b).

(d) In the case when both $u_1, u_2$ are star-nodes, we come to a similar contradiction.

(e) Let $u_1$ be a star-node $v_K$, and $u_2 \in V$; then $u_2 = s'$. This case is analogous to (a). More precisely, for an edge $sp$ in $K$ with $p \neq s'$, $(s, z_{sp}, u_1, t)$ is $G'$-shortest, implying that $(s, p, t)$ is $G$-shortest. Also $(s, s', t)$ is $G$-shortest. Take a median $q$ for $p, s', t$ in $G$. This $q$ is in $K$. Arguing as in (i), we observe that $q$ is a median for $\{p, s', y\}$ in $G'$. This implies that $z_{s'q}$ is a median for $\{u_1, u_2, y\}$.

(iii) Finally, suppose that $x \in V$, i.e., $x = s$. Up to symmetry, three cases are possible. (a) If $u_1, u_2 \in V$, take a median $q$ for $\{u_1, u_2, t\}$ in $G$. Then $q$ is a median for $u_1, u_2, y$ in $G'$ (by the argument in (i)). (b) Let both $u_1, u_2$ be split-nodes, i.e., $u_1 = z_{sp}$ and $u_2 = z_{sp'}$ for some $p, p' \in V$. Then $d(pt), d(p't) < d(st)$, and a median $q$ for $p, p', t$ in $G$ is a median for $p, p', y$ in $G'$. This implies that the star-node $v_K$ for the bi-clique $K$ including the 4-cycle $spqp's$ is a median for $u_1, u_2, y$. (c) Let $u_1$ be a split-node $z_{sp}$ and $u_2 \in V$. Then a median $q$ for $p, u_2, t$ is a median for $p, u_2, y$, whence the split-node $z' = z_{u_2q}$ is a median for $u_1, u_2, y$ (observing that $u_1$ and $z'$ are connected by a bridge-edge).

Thus, $G'$ is modular. •

Next two claims finish the proof that $G'$ is a semiframe. Let $G(x_0, \dots, x_k)$ denote the subgraph of $G$ induced by (not necessarily distinct) nodes $x_0, \dots, x_k \in V$, and similarly for $G'$.

**Claim 5.** *$G'$ has no induced subgraph $K_{3,3}^-$.*

*Proof.* We use the fact that $K_{3,3}^-$ is non-orientable and all its edges are dependent. Suppose that $G'$ has an induced subgraph $F = (X, W) \simeq K_{3,3}^-$. Then all edges of $F$ belong to the same orbit $\widetilde{Q}$ in $G'$. By Lemma 4.3, each orbit of $G'$ induced by $Q_1$ (i.e., $Q_1'$ or $Q_1''$) is orientable. Therefore, $\widetilde{Q} = Q_i'$ for some $2 \leq i \leq r$.

We know that no induced subgraph of $G$ is isomorphic to $K_{3,3}^-$, so at least one edge of $F$ is not in $G$. Also $F$ contains no split- or star-edges (since such edges belong to $Q_1'$ or $Q_1''$). These facts imply that each node of $F$ is a split-node and each edge of $F$ is a bridge-edge. Let the nodes of $F$ be numbered as shown in Fig. 2b, let a node $i$ arise by splitting an edge $u_iv_i$ of $G$, and assume that for each $ij \in W$, $u_iu_j$ and $v_iv_j$ are edges of $G$. Observe that all $u_1, \dots, u_6$ are different (e.g., $u_1 = u_6$ is impossible as $G$ is bipartite, while $u_2 = u_4$, say, would imply that the subgraph $G(u_1, u_2, v_1, v_2, v_4)$ is $K_{2,3}$ and, therefore, its edges belong to $Q_1$). So $\Gamma = G(u_1, \dots, u_6)$ includes $K_{3,3}^-$. Since $\Gamma$ is bipartite and different from $K_{3,3}^-$, $\Gamma$ is $K_{3,3}$. Similarly, $\Gamma' = G(v_1, \dots, v_k)$ is $K_{3,3}$. But then nodes 1 and 6 of $F$ must be connected by a bridge-edge; a contradiction. •

**Claim 6.** *$G'$ is hereditary modular.*

*Proof.* Suppose this is not so. Since $G'$ is modular (by Claim 4), $G'$ contains an isometric 6-cycle $C$ and, moreover, $C$ is contained in a cube $H$ in $G'$, by (ii) in Theorem 1.8. Note that if a bipartite graph $B$ contains a cube $D$ and $B$ has no induced $K_{3,3}^-$, then the subgraph of $B$ induced by the nodes of $D$ is either $D$ or $K_{4,4}$.

Let $H$ be formed by nodes $x_0, \dots, x_3, y_0, \dots, y_3$ and edges $x_ix_{i+1}$, $y_iy_{i+1}$ and $x_iy_i$, $i = 0, 1, 2, 3$ (taking indices modulo 4). Since $G'$ has no induced $K_{3,3}^-$ (by Claim 5) and $H$ is not contained in $K_{4,4}$ (as $C \subset H$ is isometric), $H$ is an induced cube in $G'$. Suppose that some node of $H$, $y_0$ say, is a star-node $v_K$. Then $x_0, y_1, y_3$ are split-nodes generated by some edges of $K$. This easily implies that $x_1, y_2, x_3$ are nodes of $K$ and that each two of them are connected by an edge, which is impossible.

So, all nodes of $H$ are only split-nodes or nodes of $G$. Suppose that some edge of $H$, $x_0y_0$ say, is a split-edge; let for definiteness $x_0 \in V$ and $y_0 = e_{x_0v_0}$. Then $x_i \in V$ and $y_i$ is a split-node $e_{x_iv_i}$ for each $i = 0, 1, 2, 3$. We observe that the subgraph $\Gamma = G(x_0, \dots, x_3, v_0, \dots, v_3)$ includes a cube. Indeed, obviously, $v_iv_{i+1} \in E$ and $v_i \neq x_{i+2}$ (otherwise $\Gamma$ is not bipartite). Also $v_0 \neq v_2$ (otherwise $G(x_0, x_1, x_2, x_3, v_0)$ is $K_{2,3}$, whence the edge $x_0x_1$ must be split). Similarly, $v_1 \neq v_3$. So $\Gamma$ is $K_{4,4}$. But then all edges of $\Gamma$ are dependent and belong to $Q$, which is impossible.

When all the nodes $x_0, \dots, x_3, y_0, \dots, y_3$ are in $V$, these induce a subgraph of $G$ including the cube $H$, and we also come to a contradiction. Finally, if all nodes of $H$ are split-nodes, then all its edges are bridge-edges. Let $x_i = z_{u_iv_i}$, and let $u_i$ be adjacent to $u_{i+1}$ in $G$, $i = 0, 1, 2, 3$. Considering $\Gamma = G(u_0, \dots, u_3, v_0, \dots, v_3)$ and arguing as above, we again observe that $\Gamma$ includes a cube $\Gamma'$ and come to a contradiction. •

By Claims 5 and 6, $G'$ is a semiframe, as required in Lemma 4.2. This completes the proof of (iii)→(ii) in Theorem 1.2. • •

## 5. Proofs of Theorems 1.4 and 1.5 and additional results

In fact, Theorem 1.4 is a by-product of the orbit splitting method described in the previous section. More precisely, let $\mu$ be a cyclically even PF-metric. By Statement 2.1, the modular closure $m$ for $\mu$ is integral. If the LG-graph $G=(V,E)$ for $m$ is a frame, i.e., all orbits of $G$ are orientable, then every time we split an orbit in question, we can choose a number $\alpha$ figured in (4.2)(iii) to be an integer. As a result, we obtain a frame $H$ such that $\mu$ is a submetric of $\mathrm{d}^H$. Then every primitive extension $\widetilde{m}$ of $\mu$ is isomorphic to a submetric of $\mathrm{d}^H$ (by the argument in the proof of (ii)$\rightarrow$(i) in Theorem 1.2 given in Section 3), i.e., $\widetilde{m}$ is integral. And if $G$ has a non-orientable orbit, then we can apply the construction to $m'=2m$. Then the first splitting of each non-orientable orbit $Q$ creates two orientable orbits $Q',Q''$ with the integer length $q/2$ of each edge, where $q$ is the length of an edge of $Q$, by Lemmas 4.3 and 4.4. So the resulting frame $H$ is such that $\mu$ is a submetric of $\mathrm{d}^H/2$ and, therefore, every primitive extension of $\mu$ is half-integral, as required.

Note that we cannot avoid the half-integrality in a general case of cyclically even PF-metrics. E.g., the modular metric $\mu=\mathrm{d}^{K_{3,3}}$ has the half-integer primitive extension $\mathrm{d}^\Gamma/2$, where $\Gamma$ is the graph obtained by splitting the orbit of $K_{3,3}$. Theorem 1.4 has the following weakened version.

**Corollary 5.1.** *If $\mu$ is an integer PF-metric, then every primitive extension of $\mu$ is quarter-integral, and it is half-integral when the LG-graph of a modular closure of $\mu$ is a frame.* $\bullet$

We cannot decrease the fractionality in this case too. $\mu=\mathrm{d}^{K_3}$ gives the simplest example when the LG-graph is a frame and $\mu$ has a non-integer primitive extension. Examples with the quarter-integrality also exist. E.g., let $T=\{s_1,\dots,s_6\}$, $\mu(s_is_j)=3$ for $1\le i<j\le 3$, and $\mu(s_is_j)=2$ otherwise ($i\ne j$). One can check that the LG-graph $G$ of the (unique) modular closure $m$ of $\mu$ is a semiframe which contains an induced subgraph $K_{3,3}$ with the length $m(e)=1/2$ of each edge $e$. Since $\mathrm{d}^{K_{3,3}}/2$ has a quarter-integer primitive extension, so does $\mu$ (in view of Statement 3.1).

Now we explain how to obtain Theorem 1.5. We rely on the following simple facts (cf. [7,8]):

(5.1) (i) if $(X',d')$ is a submetric of $(X,d)$, then $\mathcal{T}(d')\subseteq\mathcal{T}(d)$;

(ii) if $(X'',d'')$ is a tight extension of $(X,d)$, then $\mathcal{T}(d'')=\mathcal{T}(d)$.

(Property (i) is based on the observation that any tight extension $(Y,q)$ of $d'$ with $Y\cap X=X'$ can be extended to $Y\cup X$ so as to give a tight extension $\widetilde{q}$ of $d$ (by defining $\widetilde{q}(xy)=\min\{d(xs)+q(sy)\,:\,s\in X'\}$ for $x\in X$ and $y\in Y$; cf. the proof of Statement 3.1). Property (ii) follows from (i) and the fact that any tight extension of $d''$ is a tight extension of $d$, which is, e.g., easily seen from (2.1).)

As before, let $m$ be a modular closure of a given metric $\mu$, and $G=(V,E)$ the LG-graph of $m$. By Statement 2.1, $m$ is a primitive extension of $\mu$; so $m$ is tight. If $\mu$ is primitively finite, then by Theorem 1.2 $\mu$ is a submetric of $\widehat{m}=\mathrm{d}^H/\lambda$ for some frame $H$ and number $\lambda>0$. Moreover, Lemma 4.4 provides that the frame $H$ obtained by use of the orbit splitting method is such that $\widehat{m}$ is a tight extension of $m$ and, therefore, of $\mu$. Thus, $\mathcal{T}(\mu)=\mathcal{T}(\mathrm{d}^H/\lambda)$ (by (5.1)(ii)), whence the dimension of $\mathcal{T}(\mu)$ is at most two, in view of a result from [10] mentioned in the Introduction.

To see the other direction, note that if $\mu$ is not primitively finite, then some primitive (and, therefore, tight) extension of $\mu$ has a submetric $a\mathrm{d}^{K_{3,3}^-}$ with $a>0$, by the argument in Section 3. For $d=\mathrm{d}^{K_{3,3}^-}$, $\dim(\mathcal{T}(d))=3$ (this follows, e.g., from (ii) in Theorem 1.6 because $d$ admits a strictly dominating bijection, namely, that corresponding to the pairs $\{1,6\},\{2,4\},\{3,5\}$ in Fig. 2b). By (5.1), $\mathcal{T}(ad)\subseteq\mathcal{T}(\mu)$, whence $\dim(\mathcal{T}(\mu))\ge 3$, as required.

Next we derive some additional results from the above method of proof of Theorem 1.2. Throughout $\mu$ is a rational PF-metric on $T$.

**1.** The upper bound on the number of primitive extensions of $\mu$ given in Section 3 depends on the size of the frame $H$ figured in (ii) of Theorem 1.2. Under the method in Section 4, the size of $H$ (and, therefore, the bound) can grow significantly if we replace $\mu$ by $2\mu$, say, though the number of primitive extensions remains the same. The following lemma suggests a more efficient way of choosing the desired frame, which depends only on the LG-graph $G=(V,E)$ of a modular closure $m$ of $\mu$. More precisely, let $H=(W,U)$ and $d$ be the frame and metric on $W$ obtained from $G$ and $m$ by consecutively splitting the non-orientable orbits (see Remark 4.6). One can see that $H$ and $d$ do not depend on the order in which these orbits are treated.

**Lemma 5.2.** *(i) $d$ is a primitive extension of $\mu$, and (ii) every primitive extension of $\mu$ is isomorphic to a submetric of $d$.*

*Proof.* Consider a non-orientable orbit $Q$ of $G$, and let $G'=(V',E')$ and $m'$ be the semiframe and metric on $V'$ obtained from $G$ and $m$ by splitting $Q$. Then $Q$ induces one orbit $Q'$ in $G'$. Let $Z$ and $Z'$ be the sets of ends of edges in $Q$ and $Q'$, respectively. The submetric $g'=m'_{|Z'}$ is a tight extension of $g=m_{|Z}$ and all edges of $Q'$ are dependent w.r.t. 4-cycles of $G'$ meeting $Q'$. Hence, $g'$ is a primitive extension of $g$. Also $m$ is a primitive extension of $\mu$ (by Statement 2.1). So there is an extreme extension $m''$ of $\mu$ to $V'$ which coincides with $m$ on $V$ and with $g'$ on $Z'$ (by Statement 3.1). Then the tightness of $m'$ for $\mu$ (by Lemma 4.4) implies $m'=m''$. Repeatedly applying this argument to the non-orientable orbits of the currect graphs (arising from $G$ in the orbit splitting process), we obtain (i).

To see (ii), we observe that the splitting operation applied to an orientable orbit of $H$ (as well as of any other currect graph in the orbit splitting process starting with $H$) does not maintain the primitivity. More precisely, let $H' = (W', U')$ and $d'$ be obtained from $H$ and $d$ by splitting an (orientable) orbit $Q$ into two orbits $Q'$ and $Q''$ with a number $\alpha$ as defined in Lemma 4.4. Fix the orientations of $Q, Q', Q''$ as indicated in the proof of Lemma 4.3. For each split-node $z = z_{xy}$, define $\gamma_1(z) = x$ and $\gamma_2(z) = y$, where $xy \in Q$ has the orientation $(x, y)$. For each star-node $v = v_K$, define $\gamma_1(v) = s$ and $\gamma_2(v) = t$, where $s$ ($t$) is the node of $K$ whose all incident edges in $K$ are oriented from $s$ (resp. to $t$). Define $\gamma_1(x) = \gamma_2(x) = x$ for the other nodes of $H'$.

For $i = 1, 2$, let $m_i$ be the metric on $W'$ induced by $d$ and $\gamma_i$, i.e., $m_i(xy) = d(\gamma_i(x)\gamma_i(y))$ for $x, y \in W'$. It is not difficult to check that $d = \frac{q-\alpha}{q} m_1 + \frac{\alpha}{q} m_2$, where $q = d(e)$ for $e \in Q$. Thus, $d'$ is a convex combination of two extensions of $\mu$ similar to $d$. Now using induction, one can conclude that if $\overline{H}$ is the final frame with the all-unit lengths of the edges in the orbit splitting process for $(G, m)$ (assuming that $m$ is cyclically even), then $\mathrm{d}^{\overline{H}}$ is a convex combination of extensions of $\mu$ similar to $d$. This implies that each submetric of $\mathrm{d}^{\overline{H}}$ is a convex combination of metrics isomorphic to submetrics of $\mathrm{d}^H$, whence (ii) follows by the argument in the proof of (ii)→(i) of Theorem 1.2. •

**2.** Every semiframe serves as the LG-graph of a modular closure for a representative class of PF-metrics on the same set, reflecting the fact that "stretching" the edge lengths of orbits preserves the set of shortest paths. More precisely, we say that a positive function on the edges of a semiframe is *conform* if it is constant within each orbit.

**Statement 5.3.** *Let $G = (V, E)$ be a semiframe, and $\ell$ a conform function on $E$. Let $m = \mathrm{d}^{G,\ell}$. Then the sets of $G$-shortest and $m$-shortest paths in $G$ are the same. In particular, $m(e) = \ell(e)$ for all $e \in E$.*

*Proof.* Consider two paths $P$ and $P'$ with the same ends in $G$ such that $P$ is $G$-shortest. Let $Q_1, \ldots, Q_k$ be the orbits of $G$, and denote $n_i = P \cap Q_i$, $n'_i = P' \cap Q_i$ and $\ell_i = \ell(e)$ for $e \in Q_i$. We have $\ell(P) = \ell_1 n_1 + \ldots + \ell_k n_k$ and $\ell(P') = \ell_1 n'_1 + \ldots + \ell_k n'_k$. By Claim 2 in Section 4, $n_i \le n'_i$ for each $i$. Hence, $\ell(P) \le \ell(P')$. Moreover, if $|P| < |P'|$, then $n_i < n'_i$ for some $i$, whence $\ell(P) < \ell(P')$ (since $\ell$ is positive). •

(A similar property holds for arbitrary modular graphs and positive functions invariant on dependent edges (cf. [2,4]); it can be shown in a similar fashion as that in the proof of Statement 2.2.) This implies the following.

**Corollary 5.4.** *Let $G = (V, E)$ be the LG-graph of a modular closure $m$ of $\mu$. Let*

*$\ell$ be a conform function on $E$, and let $m' = \mathrm{d}^{G,\ell}$ and $\mu' = m'_{|T}$. Then $m'$ is a modular closure of $\mu'$ and $G$ is the LG-graph of $m'$.*

*Proof.* By Statements 2.2 and 5.3, $m$ and $m'$ have the same sets of shortest paths on $V$. Then the betweenness relations for $m$ and $m'$ are the same, whence $G$ is the LG-graph for $m'$. Also one can see that all steps in the process of determining the modular closure $m$ for $\mu$ remain applicable to $\mu$ and, terefore, result in $m'$ (by the primitivity). $\bullet$

**3.** Results in the above enable us to show invariantness of the modular closure of a PF-metric. To be precise, we say that extensions $(V, m)$ and $(V', m')$ of $\mu$ (resp. the LG-graphs $G = (V, E)$ for $m$ and $G' = (V', E')$ for $m'$) are the same if there is a one-to-one mapping $\omega : V \to V'$ satisfying $\omega(s) = s$ for $s \in T$ and $m(xy) = m'(\omega(x)\omega(y))$ (resp. $xy \in E \Longleftrightarrow \omega(x)\omega(y) \in E'$) for $x, y \in V$, and different otherwise.

**Theorem 5.5.** *For any PF-metric $\mu$ on $T$, the modular closure $(V, m)$ of $\mu$ is determined uniquely.*

*Proof.* Suppose there exists a modular closure $(V', m')$ of $\mu$ different from $(V, m)$. Let $G = (V, E)$ and $G' = (V', E')$ be the LG-graphs for $m$ and $m'$, respectively. Since $m$ is a primitive extension of $\mu$, it is determined uniquely by the sets of $G$-shortest $T$-paths, and similarly for $m'$ and $G'$ (in view of Statement 2.2). Therefore, $G$ and $G'$ are different. Consider three cases.

(i) Let both $G$ and $G'$ be frames, and let for definiteness $|V| \le |V'|$. By Lemma 5.2(ii), one may assume that $m'$ is a submetric of $m$. Then $|V| \le |V'|$ implies $V = V'$, whence $m = m'$; a contradiction.

(ii) Let $G'$ be a frame and $G$ not, i.e., $G$ has a non-orientable orbit. Let $H$ and $h$ be obtained from $G$ and $m$ by splitting the non-orientable orbits. By the argument in (i), one may assume that $h = m'$ and $H = G'$. Consider the metric $\nu = \mathrm{d}^G{}_{|T}$. By Lemma 5.2 and Corollary 5.4, $\mathrm{d}^G$ is a modular closure for $\nu$. Moreover, splitting the non-orientable orbits for $G$ and $\mathrm{d}^G$ makes the same frame $H = G'$ and the corresponding metric $g$ on $V$. By Statements 5.3 and 2.2, $g$ and $m'$ have the same sets of shortest paths. This implies that $g$ is a modular closure for $\nu$. So, by Statement 2.1, $g$ must be integral because $\nu$ is cyclically even (as $G$ is bipartite). But $g$ takes a non-integer value. Indeed, take an edge $e = xy$ in a non-orientable orbit of $G$. Then $e$ generates two split-edges $xz_e$ and $yz_e$ in $H$, and we have $g(xz_e) = 1/2$; a contradiction.

(iii) Let both $G, G'$ have non-orientable orbits. Let $H$ and $h$ ($H'$ and $h'$) be obtained from $G$ and $m$ (resp. $G'$ and $m'$) by splitting the non-orientable orbits. Then $H = H'$ and $h = h'$. Assume for definiteness that $|V| \le |V'|$ and consider the metric

$\nu = \mathrm{d}^G{}_{|T}$. Then $\mathrm{d}^G$ is a modular closure for $\nu$, and splitting the non-orientable orbits for $G$ and $\mathrm{d}^G$ makes the same frame $H$ and the corresponding metric $g$ on $V$. By the argument as in (ii), $f = g_{|V'}$ is a modular closure for $\nu$ and $G'$ is the LG-graph for $f$. Since $\nu$ is cyclically even (as $G$ is bipartite), $f$ is integral. Now we come to a contradiction by showing that $f$ takes a non-integer value.

Indeed, $G$ and $G'$ are different, $V \cup V'$ is in $H$ and $|V| \leq |V'|$; so there is a node $v \in V'$ which is not in $V$. Then $v$ is either a split-node or a star-node in $H$ (w.r.t. $G$). Suppose $v$ is a split-node $z_{xy}$ for an edge $xy \in E$. Since $g$ is tight for $\nu$, there are $s,t \in T$ such that $g(sx) + g(xy) + g(yt) = \nu(st)$. This implies $g(sv) = g(sx) + \frac{1}{2}$. But $g(sv) = f(sv)$ and $g(sx) = \mathrm{d}^G(sx)$, whence $f(sv)$ is not integral. Now suppose $v$ is a star-node $v_K$, where $K = (A; B)$ is a 2-clique in $G$. Take $x \in A$ and $y \in B$. Let $s,t,s',t' \in T$ be such that $g(sx) + g(xv) + g(vt) = \nu(st)$ and $g(s'y) + g(yv) + g(vt') = \nu(s't')$. Then $g(sv) = g(sx) + g(xv)$ and $g(s'v) = g(s'y) + g(yv)$, whence $f(sv) = \mathrm{d}^G(sx) + 1$ and $f(s'v) = \mathrm{d}^G(s'y) + 1$ (since $g(xv) = g(xz_{xy}) + g(z_{xy}v_K) = \frac{1}{2} + \frac{1}{2} = 1$, and similarly $g(yv) = 1$). This and the evenness of $f(sv) + f(s'v) + f(ss')$ imply that $\mathrm{d}^G(sx) + \mathrm{d}^G(s'y) + f(ss')$ is even. But $f(ss') = \mathrm{d}^G(ss')$ and $\mathrm{d}^G(xy) = 1$; so $\mathrm{d}^G(sx) + \mathrm{d}^G(xy) + \mathrm{d}^G(ys') + \mathrm{d}^G(ss')$ is odd; a contradiction. •

We can summarize the above results as follows.

**Corollary 5.6.** *For any PF-metric $\mu$, there are unique modular closure $m$ of $\mu$, LG-graph $G$ of $m$, and pair $(H,h)$, where $H = (W,U)$ is a frame and $h$ is a conform function on $U$ such that $\mathrm{d}^{H,h}$ is a primitive extension of $\mu$ and every primitive extension of $\mu$ is isomorphic to a submetric of $\mathrm{d}^{H,h}$. The pair $(H,h)$ is formed from $(G,m)$ by splitting the non-orientable orbits of $G$.* •

We call $G$ and $H$ in this corollary the *canonical* semiframe and frame for $\mu$, respectively (possibly $G = H$) and call $(H,h)$ the *generator of primitive extensions* of $\mu$. It is tempting to hope that *every* submetric $(W',g')$ of $\mathrm{d}^{H,h}$ with $T \subseteq W' \subseteq W$ is primitive for $\mu$ (which would imply the exact formula $|\Pi(\mu)| = 2^{|W|-|T|}$). However, this need not hold. E.g., let $G = (V,E) \simeq K_{3,3}$ and $\mu = \mathrm{d}^G$. Then $H = (W,U)$ is obtained by splitting of the orbit of $G$ and $h$ is identically $1/2$. One can see that for any $e \in E$, the submetric of $\mathrm{d}^{H,h}$ on $W' = V \cup \{z_e\}$ is the half-sum of two metrics on $W'$ similar to $\mathrm{d}^G$.

**4.** As mentioned in the Introduction, [10] gives an explicit combinatorial construction of the tight span $\mathcal{T}(\mathrm{d}^H)$ for an arbitrary frame $H$. Using the orbit splitting method, we can generalize that construction to the metrics $\mathrm{d}^{H,h}$, where $H$ is a frame and $h$ is a rational conform function on its edges, and, as a result, describe the "canonical structure" of the tight span of a PF-metrics $\mu$ (such a structure looks somewhat different from the polyhedral structure of tight spans mentioned in the Introduction).

First we recall the construction of $\mathcal{T}(\mathrm{d}^H)$ for a frame $H=(W,U)$. Each edge $e$ of $H$ is regarded as being homeomorphic to the closed interval (segment) $[0,1]\subset\mathbb{R}^1$ with the natural metric $\sigma^e$ on it. Each 4-cycle $C=v_0v_1v_2v_3v_0$ (considered up to reversing and cyclically shifting) is expanded into a 2-dimensional disc $D^C$. Formally, $D^C$ is homeomorphic to $[0,1]\times[0,1]\subset\mathbb{R}^2$, the nodes $v_0,v_1,v_2,v_3$ are identified with the points (0,0),(0,1),(1,1),(1,0), respectively, and the edges with the corresponding segments. $D^C$ is endowed with the $\ell_1$-metric $\sigma^C=\sigma^{v_0v_1}\oplus\sigma^{v_1v_2}$, i.e., for points $x=(\xi,\eta)$ and $y=(\xi',\eta')$ in $D^C$, $\sigma^C(xy)=|\xi-\xi'|+|\eta-\eta'|$. If two 4-cycle $C=v_0v_1v_2v_3v_0$ and $C'=u_0u_1u_2u_3u_0$ have three common nodes, $v_i=u_i$ for $i=0,1,2$ say, we identify the corresponding halves (triangles) in $D^C$ and $D^{C'}$; namely, assuming for definiteness that $v_0,v_1,v_2$ are represented as (0,0),(0,1),(1,1) in both discs, respectively, we identify each point $(\xi,\eta)$ for $0\le\xi\le\eta\le1$ in $D^C$ with the $(\xi,\eta)$ in $D^{C'}$. As a result, every bi-clique $K=(A;B)$ with $A=\{s_1,s_2\}$ and $B=\{t_1,\ldots,t_k\}$ produces the shape $F(K)$, called the *folder* of $K$, homeomorhic to the space formed by sticking together $k$ copies of the triangle $\{(\xi,\eta)\,:\,0\le\xi\le\eta\le1\}$ along the side $\{(\alpha,\alpha)\,:\,0\le\alpha<1\}$; see Fig. 8 for $k=5$. The above metrics $\sigma^C$ for 4-cycles $C$ in $K$ give the metric $\sigma^K$ on $F(K)$. In view of Statement 4.1, two different folders have at most one vertex or one edge in common.

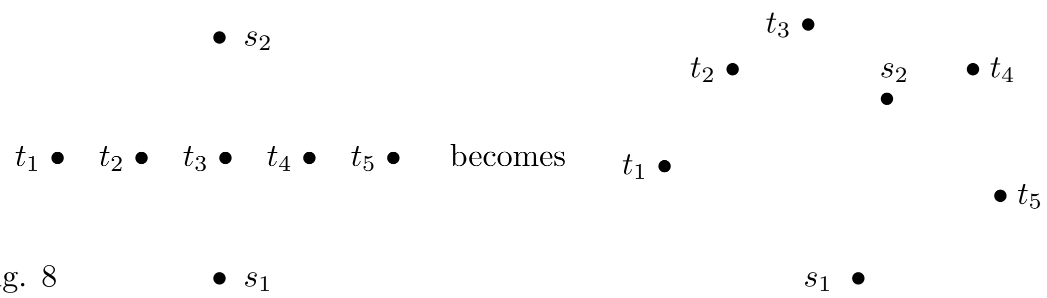


Fig. 8

The resulting space is just $\mathcal{T}(\mathrm{d}^H)=(\mathcal{X},\sigma)$, where $\mathcal{X}=\cup(F(K)\,:\,K\in\mathcal{K}(H))$ and the global metric $\sigma$ on $\mathcal{X}$ is defined in a natural way: for $x,y\in\mathcal{X}$, $\sigma(xy)$ is the infimum of values $\sigma^{q_1}(x_0x_1)+\ldots+\sigma^{q_r}(x_{r-1}x_r)$ over all finite sequences $x=x_0,x_1,\ldots,x_r=y$ in which each two $x_{i-1},x_i$ occur in the folder $F(q_i)$ of a bi-clique $q_i$ or in a bridge $q_i$.

Now suppose that $h$ takes value 2 on the edges of some orbit $Q$ of $H$, and 1 on the other edges of $H$. Let $H'$ be obtained by splitting the orbit $Q$, taking $\alpha=1$ (see Lemma 4.4). Since $\mathrm{d}^{H'}$ is a tight extension of $g=\mathrm{d}^{H,h}$, we have $\mathcal{T}(g)=\mathcal{T}(\mathrm{d}^{H'})$; so the folder structure of $\mathcal{T}(\mathrm{d}^{H'})$ describes $\mathcal{T}(g)$. However, we can describe $\mathcal{T}(g)=(\widehat{\mathcal{X}},\widehat{\sigma})$ in terms of $H$ and $h$ themselves, as follows.

(i) Let $K=xyuvx$ be a simple bi-clique (4-cycle) in $\mathcal{K}(Q)$ with $xy,uv\in Q$ (see the definition in the beginning of Section 4). Then $K$ induces two bi-cliques (4-cycles) $K'=xzz'vx$ and $K''=zyuz'z$ in $H'$, where $z=z_{xy}$ and $z'=z_{uv}$. Each of $F(K'),F(K'')$ is represented as the square $[0,1]\times[0,1]$, and the common segment between $z$ and $z'$ sticks them together into a rectangle of size $2 \times 1$. In other words, one may think that the region of $\widehat{\mathcal{X}}$ spanned by $K$ is again $F(K) = D^K$ but now with the "stretched" metric $\sigma' = (2\sigma^{xy}) \oplus \sigma^{yv}$, defined by $\sigma'((\xi,\eta)(\xi',\eta')) = |\xi - \xi'| + 2|\eta - \eta'|$ (letting $x = (0,0)$ and $y = (1,0)$).

(ii) Let $K = (A; B)$ be a non-simple bi-clique in $\mathcal{K}(Q)$. Then $h(e) = 2$ for all edges $e$ of $K$, and $K$ induces $|A| + |B|$ bi-cliques $K_a = (\{a, v_K\}; \{z_{ab} : b \in B\})$, $a \in A$, and $K_b = (\{b, v_K\}; \{z_{ab} : a \in A\})$, $b \in B$, in $H'$. One can see that the union $F'$ of folders $F(K_s)$, $s \in A \cup B$, is naturally homeomorphic to $F(K)$, but the metric $\sigma'$ on $F'$ is twice as much stretched, i.e., $\sigma' = 2\sigma^K$.

In case of a general rational conform function $h$ on $U$, we can apply similar arguments (or use induction) to conclude with the following.

**Theorem 5.7.** *Let $\mu$ be a PF-metric, and $(H, h)$ the generator of primitive extensions of $\mu$. Let $\mathcal{T}(\mu) = (\widehat{\mathcal{X}}, \widehat{\sigma})$ and $\mathcal{T}(\mathrm{d}^H) = (\mathcal{X}, \sigma)$. Then $\widehat{\mathcal{X}} = \mathcal{X}$ and $\widehat{\sigma}$ is the globalization of the metrics $\widehat{\sigma}^K$ on folders $F(K)$ of bi-cliques and the metrics $\widehat{\sigma}^e$ on bridges $e$ of $H$, where: (i) for each simple bi-clique $K$ with edges $e, e'$ in different orbits, $\widehat{\sigma}^K = (h(e)\sigma^e) \oplus (h(e')\sigma^{e'})$; (ii) for each non-simple bi-clique $K$, $\widehat{\sigma}^K = h(e)\sigma^K$ ($e$ is an edge of $K$); and (iii) for each bridge $e$, $\widehat{\sigma}^e = h(e)\sigma^e$.* •

**Remark 5.8.** Theorem 5.7 describes the structure of $\mathcal{T}(\mu)$ in terms of the canonical frame. In a similar fashion, we can represent $\mathcal{T}(\mu)$ by use of the canonical semiframe $G = (V, E)$ and length function $g = m_{|E}$, where $m$ is the modular closure of $\mu$. To this purpose, we observe that for each bi-clique $K = (A; B)$ with $|A|, |B| \geq 3$ in $G$, the space $\mathcal{T}(\mathrm{d}^K) = (F(K), \sigma^K)$ is again obtained by identifying the corresponding halves of discs $D^C$ and $D^{C'}$ for all 4-cycles $C$ and $C'$ in $K$ with three common nodes; this can be shown by considering the frame $H(K)$ obtained from $K$ by splitting its only orbit and using the fact that $\mathcal{T}(\mathrm{d}^K) = \mathcal{T}(\frac{1}{2}\mathrm{d}^{H(K)})$. By analogy, we call such an $F(K)$ the folder of $K$ (now it has a more complicated facet structure; in particular, it is not homeomorphically embeddable in $\mathbb{R}^3$). Then Theorem 5.7 remains valid with $(G, g)$ in place of $(H, h)$.

An interesting question is how the above folder structure of $\mathcal{T}(\mu)$ is related to its polyhedral structure defined in [7,8].

## 6. A relationship to multicommodity flows

Results on PF-metrics find applications to one sort of multiflow (multicommodity flow) problems. An instance of this problem is given by an (undirected) graph $G =$

$(V, E)$, a *capacity* function $c : E \to \mathbb{R}_+$ on its edges, and a metric $\mu$ on a subset $T \subseteq V$ (of which nodes are called *terminals*). A ($c$-admissible) *multiflow* $f$ consists of $T$-paths $P_1, \ldots, P_k$ in $G$ along with nonnegative real weights $\lambda_1, \ldots, \lambda_k$ satisfying the capacity constraint

$$f^e := \sum (\lambda_i \ : \ P_i \ \text{ contains } e) \leq c(e) \quad \text{for all } \ e \in E \tag{6.1}$$

(as before, a $T$-path is a simple path connecting distinct $s, t \in T$). For $s, t \in T$, define $\mathrm{val}(f_{st}) = \sum(\lambda_i \ : \ P_i$ connects $s$ and $t$), the *value* of the flow $f_{st}$ in $f$ between terminals $s$ and $t$. The *gain* of each $f_{st}$ is defined to be $\mu(st)\mathrm{val}(f_{st})$.

In the *metric-weighted maximum multiflow problem* (briefly, MMP), one is asked to maximize the total gain $\langle \mu, f \rangle := \sum(\mu(st)\mathrm{val}(f_{st}) \ : \ s, t \in T)$ of a multiflow $f$. This is a linear program and, assigning dual variables $\ell(e)$, $e \in E$, to the inequalities in (6.1), we can consider the dual linear program that consists in minimizing $c\ell = \sum(c(e)\ell(e) \ : \ e \in E)$ over the "length functions" $\ell : E \to \mathbb{R}_+$ such that the length $\ell(P)$ of any $T$-path $P$ connecting terminals $s, t$ is at least $\mu(st)$. Since $c$ is nonnegative, we can replace $\ell(e)$ by $\mathrm{d}^{G \cup K_T, \ell}(e)$ for $e \in E \cup E_T$, letting $\ell(st) = \mu(st)$ for $s, t \in T$. Then the dual problem becomes equivalent to the following:

(6.2) minimize $cm$ over all extensions $m$ of $\mu$ to $V$

(assuming w.l.o.g. that $c(xy) = 0$ if $x, y \in V$ and $xy \notin E$). So $\max\{\langle \mu, f \rangle\} = \min\{cm\}$, where $f$ ranges the multiflows for $G, c, T$ and $m$ ranges the extensions of $\mu$ to $V$. This relation occurred in [11] where other properties and applications of MMP are also shown.

Note that if $m, m', m''$ are extensions of $\mu$ to $V$ such that $m \geq \lambda m' + (1 - \lambda)m''$ for some $0 \leq \lambda \leq 1$, then $cm \geq \lambda cm' + (1 - \lambda)cm''$ (since $c \geq 0$), implying $cm \geq \min\{cm', cm''\}$. Therefore, an optimal solution to (6.2) is achieved by some extreme extension $m$ of $\mu$ to $V$. On the other hand, by linear programming arguments, every extreme extension $m$ occurs as a unique optimal solution to (6.2) for a certain capacity function $c$ on $E_V$. (One can give a direct construction of such a $c$ as follows. Let $V' \subseteq V$ be a maximal subset with $m_{|V'}$ positive, let $E' = \{e \in E_V \ : \ m(e) = 0\}$, and let $P_1, \ldots, P_k$ be all $m$-shortest $T$-paths in $K_{V'}$. Then the multiflow $f$ formed by these paths taken with unit weights is optimal for the capacities $c(e) = f^e$ for $e \in E_V - E'$ and $c(e) = 1$ for $e \in E'$. Moreover, these paths must be $m$-shortest for any optimal solution $m$ to (6.2) for this $c$. Then $m$ is determined uniquely on $V'$ (since $m$ is extreme), whence $m$ is so on $V$ (since $f^e = 0 < c(e)$ for $e \in E'$ implies $m(e) = 0$).)

Thus $\Pi(\mu)$ gives the minimal list of (unavoidable) positive metrics whose similar metrics occur as optimal dual solutions to MMP among all graphs $G = (V, E)$ with $V \supseteq T$ and capacities $c$ on $E$. Roughly speaking, for $\mu$ fixed, MMP admits a finite number of "types" of optimal dual solutions if and only if $\mu$ is primitively finite. Also Theorem 1.4 implies the following.

**Corollary 6.1.** *If $\mu$ is a cyclically even PF-metric, then (6.2) has an integer optimal solution.*

*Example.* If $|T| \geq 3$ and $\mu(st) = 1$ for all distinct $s, t \in T$ (i.e., $\mu = \mathrm{d}^{K_T}$), then $\mu$ has, besides itself, only one primitive extension $m = \frac{1}{2}\mathrm{d}^H$, where $H = (W, U)$ is the graph with $W = T \cup \{v\}$ and $U = \{sv \,:\, s \in T\}$. This gives the well-known minimax relation proved in [5,15] (originally stated in [13]): for any $G, T, c$ as above, the maximum total value of a multiflow consisting of flows between arbitrary pairs of distinct terminals is equal to $\frac{1}{2}\sum(q_s \,:\, s \in T)$, where $q_s$ is the maximum capacity of a cut in $(G, c)$ separating $s$ and $T - \{s\}$.

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