Multiflows and Disjoint Paths of Minimum Total Cost

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Abstract. In this paper we discuss a number of recent and earlier results in the field of combinatorial optimization that concerns problems on minimum cost multiflows (multicommodity flows) and edge-disjoint paths. More precisely, we deal with an undirected network N consisting of a supply graph G, a commodity graph H and nonnegative integer-valued functions of capacities and costs of edges of G, and consider the problems of minimizing the total cost among (i) all maximum multiflows, and (ii) all maximum *integer* multiflows.

For problem (i), we discuss the denominators behavior in terms of H. The main result is that if H is complete (i.e. flows between any two terminals are allowed) then (i) has a *half-integer* optimal solution. Moreover, there are polynomial algorithms to find such a solution. For problem (ii), we give an explicit combinatorial minimax relation in case of H complete. This generalizes a minimax relation, due to Mader and, independently, Lomonosov, for the maximum number of edge-disjoint paths whose ends belong to a prescribed subset of nodes of a graph. Also there exists a polynomial algorithm when the capacites are all-unit.

The minimax relation for (ii) with H complete allows to describe the dominant for the set of (T,d)-joins (extending the notion of T-join) and the dominant for the set of maximum multi-joins of a graph. Also other relevant results are reviewed and open questions are raised.

Keywords. Multicommodity Flow, Disjoint Paths, Dominant

Abbreviated title. Multiflows and disjoint paths.

1. Definitions, problems, results

Suppose we are given nodes $s_1,\dots,s_k,t_1,\dots,t_k$ in a graph G, and we wish to find pairwise edge-disjoint paths $P_1,\dots,P_r$ such that: (i) each P_i connects s_j and t_j for some j; (ii) the number r of paths is as large as possible; and (iii) the sum of lengths of these paths is as small as possible, subject to (i),(ii). When can this problem be efficiently solved? This problem is known to be NP-hard even if $k=2$ and condition (iii) is dropped [10]. On the other hand, it turns out that the desired paths can be found

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in polynomial time if the pairs $\{s_1, t_1\}, \ldots, \{s_k, t_k\}$ form (the edge-set of) a complete graph. The latter result is one of those surveyed in this paper.

We start with some definitions and conventions. Throughout, unless otherwise is explicitly stated, by a *graph* we mean an undirected graph without multiple edges and loops; VG and EG denote the node-set and edge-set of a graph G. An edge with end nodes u and v is denoted by uv.

We deal with a network $N = (G, H, c, a)$ consisting of a *supply* graph G, a *commodity* graph H with $VH \subseteq VG$, a *capacity* function $c : EG \to \mathbb{Z}_+$ and a *cost* function $a : EG \to \mathbb{Z}_+$ ($\mathbb{Z}_+$ is the set of nonnegative integers). The edges of H indicate the pairs of nodes of G that are allowed to connect by flows.

Let $\mathcal{P} = \mathcal{P}(G, H)$ be the set of simple paths in G connecting nodes s and t for $st \in EH$. By a (c-*admissible*) multicommodity flow, or, briefly, a *multiflow*, we mean a nonnegative rational-valued function $f : \mathcal{P} \to \mathbb{Q}_+$ satisfying the capacity constraint

$$(1.1) \qquad \zeta^f(e) := \sum (f_P \,:\, e \in P \in \mathcal{P}) \le c(e) \ \text{ for all } \ e \in EG$$

(considering a path as an edge-set). We call $\sum(f_P \,:\, P \in \mathcal{P})$ the *total value* of f and denote it by $\mathrm{val}(f)$. Let $\nu^* = \nu^*(G, H, c)$ be the maximum total value of a multiflow; f is called *maximum* if $\mathrm{val}(f) = \nu^*$. Similarly, considering the set of (c-admissible) integer multiflows $f : \mathcal{P} \to \mathbb{Z}_+$, we define a *maximum integer multiflow* and the number $\nu = \nu(G, H, c)$. Clearly $\nu \le \nu^*$.

We also associate with a multiflow f its *total cost* a_f that is $\sum(a_e \zeta^f(e) \,:\, e \in EG)$, or $\sum(a(P) f_P \,:\, P \in \mathcal{P})$, where $a(P)$ is the cost of P. [For a function $g : S \to \mathbb{R}$ and a subset $S' \subseteq S$, $g(S')$ stands for $\sum(g_e \,:\, e \in S')$.] Two problems are discussed:

(1.2) *Find a maximum multiflow f with a_f as small as possible*;

(1.3) *Find a maximum integer multiflow f with a_f as small as possible.*

Thus, (1.3) is the integer strengthening of (1.2), while (1.2) is the fractional relaxation of (1.3). We refer to (1.2) ((1.3)) as the *fractional* (resp. *integer*) problem. One may assume that H has no isolated (i.e., zero degree) nodes; VH is called the set of *terminals* of the network and denoted by T. A path in G connecting two distinct terminals is called a *T-path*.

When $a = \mathbb{0}$, we obtain the "pure" maximum and maximum integer multiflow problems. When $c = \mathbb{1}$ and $a = \mathbb{1}$, (1.3) turns into the above-mentioned problem on edge-disjoint paths of minimum total length. The examples below demonstrate some properties of (1.2) and (1.3) depending on the graph H. Here and later on for $X \subseteq VG$, $\delta(X) = \delta^G(X)$ denotes the set of edges of G with exactly one end in X, called the *cut* in G induced by X. We say that $\delta(X)$ *separates* nodes u and v (or sets $Y, Z \subset VG$) if one of them is (entirely) contained in X and the other in $VG - X$. If $c(\delta(\{v\}))$ is even for each $v \in VG - T$, c is called *inner Eulerian.*

Example 1. H has a unique edge st. Then (1.2) is the undirected min-cost max-flow problem, and by a classic result in network flow theory, $\nu^* = \nu$ and ν equals the minimum capacity $c(\delta(X))$ of a cut $\delta(X)$ separating s and t (see, e.g., [12]). Moreover, plenty of polynomial and strongly polynomial algorithms are known to solve (1.3) (see [1,14] for a survey).

Example 2. $T = \{s, t, s', t'\}$ and $EH = \{st, s't'\}$. In case $a = \mathbb{0}$, (1.2) turns into the (undirected) maximum two-commodity flow problem, and it has a *half-integer* optimal solution (o.s.) [17] (or an integer o.s. in the inner Eulearian case [34]). However, we shall see later that for a general a, one cannot guarantee that (1.2) has an o.s. with bounded denominators (when G varies). In its turn, (1.3) is NP-hard even if $a = \mathbb{0}$ and $c = \mathbb{1}$ [10].

Example 3. H is the complete graph K_T with node-set T, and $|T| \geq 3$. In other words, flows connecting any two distinct terminals are allowed. We refer to a multiflow for G, K_T, c as a T-*multiflow*. Lovász [28] and, independently, Cherkassky [4] established two results on T-multiflows. First, $2\nu^*$ is equal to the sum over $s \in T$ of the minimum capacities of cuts separating s and $T - \{s\}$ (this minimax relation was originally stated in [26]). Second, if c is inner Eulerian then there exists a maximum T-multiflow which is integer-valued; so there exists a half-integer maximum T-multiflow for arbitrary (integral) capacities. Moreover, such a multiflow can be found in strongly polynomial time [4,18] (in [18] this is reduced to solving $\log |T|$ maximum flow problems). The maximum integer T-multiflow problem turned out to be much more complicated. Mader [31] and, independently, Lomonosov [27] found a minimax relation involving ν, which can be expressed as

$$\nu = \frac{1}{2} \min\{\sum_{s \in T} c(\delta(Y_s)) - \eta\}, \tag{1.4}$$

where the minimum is taken over the collections $\{Y_s : s \in T\}$ of pairwise disjoint sets $Y_s \subset VG$ with $Y_s \cap T = \{s\}$, and η is the number of components K with $c(\delta^G(VK))$ odd that appear when the Y_s's are removed from G.

We now outline results on problems (1.2) and (1.3) discussed in this paper. The case $H = K_T$ will be most important.

1. A reasonable question arises: given a natural number k, does there exist a rational optimal solution to (1.2) with the denominators of all components not exceeding k? It seems to be a difficult task when we deal with an individual problem (1.2). Nevertheless, it turned out that this can be answered in terms of commodity graph H. For H fixed, define $\varphi(H)$, the *fractionality* of problem (1.2) with respect to H, to be the minimum natural number k such that for any network (G, H, c, a) problem (1.2) has an o.s. f for which kf is integer-valued. If such a k does not exist, we say that H has *unbounded fractionality*, denoting this as $\varphi(H) = \infty$.

For example, $\varphi(H)=1$ if $|EH|=1$. More generally, $\varphi(H)=1$ for any complete bipartite graph H, by the multi-terminal version of the min-cost max-flow problem [12]. On the other hand, it is easy to show that $\varphi(H)\ge 2$ for all other graphs H. The next result is less trivial: if $H=K_T$ then (1.2) has a *half-integer* o.s. [19]; hence, $\varphi(K_T)=2$ if $|T|\ge 3$. This fact was proved by considering the following slightly more general *parameteric problem* which combines both objectives figured in (1.2):

(1.5) *given* $p\in \mathbb{Q}_+$, *maximize the linear objective function* $p\,\mathrm{val}(f)-a_f$ *among all multiflows* f *for* G, K_T, c.

Obviously, (1.5) becomes equivalent to (1.2) when p is large enough. The above-mentioned result is an immediate corollary from the following theorem.

Theorem 1 [19]. *If $H=K_T$ then for any $p\in\mathbb{Q}_+$ problem (1.5) has a half-integer optimal solution f.*

As a consequence, we observe that $\varphi(H)=2$ for any *complete multi-partite* graph H with $k\ge 3$ parts (i.e., VH admits a partition $\{T_1,\dots,T_k\}$ such that $\{s,t\}\in EH$ if and only if $s\in T_i$ and $t\in T_j$ for $i\ne j$). For we can add to G new nodes $t_1,\dots,t_k$ and edges t_is ($s\in T_i$) with the same rather large capacities and costs; then any o.s. for the resulting network with the complete graph on $\{t_1,\dots,t_k\}$ as commodity graph yields an o.s. for the original network. On the other hand, the following is true.

Theorem 2 [20]. *If H is not complete multi-partite then $\varphi(H)=\infty$.*

This theorem is reduced to examination of few instances of H because of the following simple fact.

Statement 1.1. *If H' is an induced subgraph of H then $\varphi(H')\le\varphi(H)$.*

Proof. Given a network $N'=(G',H',c',a')$, add to G' the elements $s\in VH-VH'$ as isolated nodes and denote the resulting network by N. Then N and N' have the same sets of optimal solutions, whence the result follows. •

There are exactly three minimal, under taking induced subgraphs, graphs that are not complete multi-partite, namely, H_1,H_2,H_3 drawn in Fig. 1. Hence, by Statement 1.1, it suffices to show that $\varphi(H_i)=\infty$, $i=1,2,3$. We explain why the fractionality for these H_i's is unbounded in Section 3.

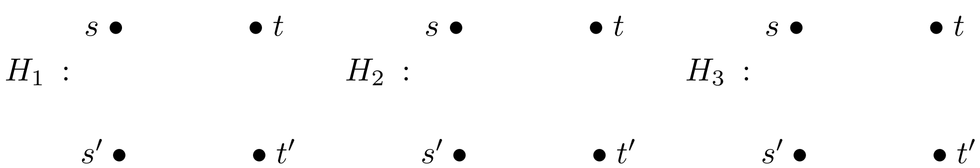

Fig. 1

2. The program dual of (1.5) can be written as

(1.6) minimize $c\gamma$ subject to

$$\gamma \in \mathbb{Q}_+^{EG} \;\text{ and }\; \mathrm{dist}_{a+\gamma}(s,t) \geq p \;\text{ for all }\; s,t \in T,\; s \neq t,$$

where for $\ell : EG \to \mathbb{Q}_+$, $\mathrm{dist}_\ell(u,v)$ denotes the ℓ-*distance* between nodes u and v, i.e., the minimum ℓ-length $\ell(P)$ of a path P in G that connects u and v.

Example 4. Let G be as in Fig. 2a, $T = \{s_1, \ldots, s_6\}$, $c = \mathbb{1}$ and $a = \mathbb{1}$. There is an only optimal T-multiflow, namely, that takes value 1/2 on the six paths shown in Fig. 2b, and zero on the other T-paths. Suppose $p = 7$. Then an optimal γ to (1.6) is zero on the edge uv and 2.5 on the other edges.

Fig. 2 (a) (b)

The original proof of Theorem 1 given in [19] was constructive and provided by a pseudo-polynomial algorithm. Being within frameworks of the primal-dual linear programming method, this algorithm is based on a parametric approach, like that used in the classic algorithm of Ford and Fulkerson [12] for the min-cost max-flow problem, but now in a more complicated context. In fact, it finds optimal primal and dual solutions simultaneously for all $p \in \mathbb{Q}_+$. More precisely, it constructs, step by step, a sequence $0 = p_0 \leq p_1 < p_2 < \ldots < p_M$ of rationals, a sequence $f_0, f_1, \ldots, f_M$ of half-integer T-multiflows and a sequence $\gamma_0, \gamma_1, \ldots, \gamma_M, \gamma_{M+1}$ of functions on EG such that: (i) for $i = 0, \ldots, M-1$ and $0 \leq \varepsilon \leq 1$, f_i and $(1-\varepsilon)\gamma_i + \varepsilon\gamma_{i+1}$ are o.s. to (1.5) and (1.6) with $p = (1-\varepsilon)p_i + \varepsilon p_{i+1}$, respectively; and (ii) for $0 \leq \varepsilon < \infty$, f_M and $\gamma_M + \varepsilon\gamma_{M+1}$ are o.s. to these programs with $p = p_M + \varepsilon$. In particular, f_M is a maximum T-multiflow.

The key idea in [19] is that, at each iteration, the new optimal f and γ can be obtained by solving the usual maximum flow problem in a certain "skew-symmetric" digraph, called a *double covering* over G. A shorter, though non-algorithmic, proof of Theorem 1 is described in [21]; it is also based on double covering techniques. We outline this proof in Section 2.

Two more results were obtained in [21]. It was shown that the dual program (1.6) has a half-integer o.s. whenever p is an integer. Also a strongly polynomial algorithm to find a half-integer o.s. to (1.2) with $H = K_T$ was developed there. However, this algorithm is not "purely combinatorial" as it uses the ellipsoid method.

Recently Goldberg and the author [13] designed two polynomial algorithms for finding a half-integer o.s. to (1.2) with $H = K_T$. Both algorithms are combinatorial and they handle within the original graph G itself rather than double coverings. One of these applies scaling on capacities, while the other scaling on costs (cf. [11,6] and [33,2] for the min-cost max-flow problem).

3. Apparently more significant results in the area we discuss were recently obtained for the integer problem (1.3) with $H = K_T$. Without loss of generality we assume that the capacities are all-unit (since, for an arbitrary $c \in \mathbb{Z}_+^{EG}$, splitting each edge e into c_e parallel edges of the same cost a_e makes an equivalent problem). As before, it is convenient to deal with a parameteric problem, namely,

(1.7) *given $p \in \mathbb{Q}_+$, maximize the objective function $\psi(p, \mathcal{D}) = p|\mathcal{D}| - a(\mathcal{D})$ among all sets $\mathcal{D}$ of pairwise edge-disjoint T-paths in G,*

where $a(\mathcal{D})$ stands for $\sum(a(P) \,:\, P \in \mathcal{D})$. Note that the objective of (1.6) gives an upper bound to $\psi(p, \mathcal{D})$, namely, $\psi(p, \mathcal{D}) \le \gamma(EG)$ for any γ as in (1.6). Simple examples show that there can be a gap between $\max\{\psi(p, \mathcal{D})\}$ and $\min\{\gamma(EG)\}$. Nevertheless, one can modify γ so as to get an exact upper bound. This gives an explicit combinatorial minimax relation involving $\psi(p, \mathcal{D})$.

To state it, we need some terminology. A set of pairwise edge-disjoint T-paths is referred as a *packing*. A pair $\phi = (X_\phi, U_\phi)$ is called an *inner fragment* if $X_\phi \subseteq VG - T$, $U_\phi \subseteq \delta(X_\phi)$, and $|U_\phi|$ is odd. Let $\mathcal{F}^0$ denote the set of inner fragments. The *characteristic function* χ_ϕ of ϕ is defined on EG by

$$\begin{aligned} \chi_\phi(e) &= 1 \quad \text{if } e \in U_\phi, \\ &= -1 \quad \text{if } e \in \delta(X_\phi) - U_\phi, \\ &= 0 \quad \text{otherwise.} \end{aligned} \tag{1.8}$$

Given $\beta : \mathcal{F}^0 \to \mathbb{R}_+$ and $\gamma : EG \to \mathbb{R}_+$, define the function $\ell = \ell^{\beta,\gamma}$ on EG as

$$\ell = a + \gamma + \sum(\beta_\phi \chi_\phi \,:\, \phi \in \mathcal{F}^0) \tag{1.9}$$

We say that (β, γ) is *p-admissible* if:

(1.10) $\ell^{\beta,\gamma}$ is nonnegative;

(1.11) $\mathrm{dist}_{\ell^{\beta,\gamma}}(s, t) \ge p$ for all distinct $s, t \in T$.

Theorem 3 [22]. *For any $p \ge 0$,*

$$\max\{\psi(p, \mathcal{D})\} = \min\{\gamma(EG) + \sum(\beta_\phi(|U_\phi| - 1) \,:\, \phi \in \mathcal{F}^0)\}, \tag{1.12}$$

where $\mathcal{D}$ ranges all packings and (β, γ) ranges all p-admissible pairs.

For instance, if G, T, c, a, p are as in Example 4, then any o.s. $\mathcal{D}$ to (1.7) consists of three T-paths covering all edges of G. The equality in (1.12) is achieved by assigning $\beta_{\phi_1} = \beta_{\phi_2} = 1/2$, $\gamma_{uv} = 0$ and $\gamma_e = 2$ for the other six edges e of G, where ϕ_1, ϕ_2 are the inner fragments with $X_{\phi_1} = \{u\}$, $U_{\phi_1} = \{us_i \,:\, i = 1, 2, 3\}$, $X_{\phi_2} = \{v\}$ and $U_{\phi_2} = \{vs_i \,:\, i = 4, 5, 6\}$.

The inequality max $\le$ min in (1.12) is easy. Indeed, for a packing $\mathcal{D}$ and a p-admissible (β,γ) we have:

$$
\begin{aligned}
(1.13)\qquad\qquad \psi(p,\mathcal{D}) &= \sum_{P\in\mathcal{D}}(p-a(P)) \\
&\le \sum_{P\in\mathcal{D}}\Big(\gamma(P)+\sum_{\phi\in\mathcal{F}^0}\beta_\phi\chi_\phi(P)\Big) \qquad \text{(by (1.9) and (1.11))} \\
&\le \gamma(EG)+\sum_{\phi\in\mathcal{F}^0}\beta_\phi\sum_{P\in\mathcal{D}}\chi_\phi(P) \qquad \text{(as the paths in } \mathcal{D} \text{ are edge-disjoint)} \\
&\le \gamma(EG)+\sum_{\phi\in\mathcal{F}^0}\beta_\phi(|U_\phi|-1) \quad \text{(as } |U_\phi| \text{ is odd while } \chi_\phi(P) \text{ is even).}
\end{aligned}
$$

The proof of equality in (1.12) is more sophisticated and it uses numerous combinatorial arguments; a sketch of this proof is outlined in Section 4. Note that the proof is constructive and it gives rise to a strongly polynomial algorithm to solve (1.7), or a pseudo-polynomial algorithm for an arbitrary c. In particular, it gives a polynomial algorithm to compute $\nu(G,K_T,\mathbb{1})$. As shown in [22], the optimal (β,γ) found at the last iteration can be transformed into cuts $\delta(Y_s)$ $(s\in T)$ occurred in (1.4).

Relation (1.12) and the algorithm can be obviously extended to a complete multipartite H. On the other hand, the following is true.

Theorem 4. *If H is not complete multi-partite, then (1.3) is NP-hard even for $c=\mathbb{1}$ and $a=\mathbb{0}$.*

To see this, it suffices to examine the only graphs $H=H_1,H_2,H_3$ as in Fig. 1. The theorem for H_1 is just the corresponding result in [10]. This implies the theorem for H_2 and H_3. Indeed, given a natural k, the problem to decide whether $k\le\nu(G,H_1,\mathbb{1})$ is obviously reduced to computing $\nu(G',H'_i,\mathbb{1})$ for $i=2$ or $i=3$, where G' is formed from G by adding nodes $t_1,\dots,t_4$ and k parallel edges connecting s_j and t_j $(j=1,2,3,4)$, and H'_i is the graph with $VH'_i=\{t_1,\dots,t_4\}$ and $EH'_i=\{t_jt_q\,:\,s_js_q\in EH_i\}$.

4. Theorem 3 has interesting applications. Suppose we are given a graph G', a set $T'\subseteq VG'$ and a function $d:T'\to\mathbb{Z}_+$ (of *demands*) such that $d(T')$ is even. We call a subset $B'\subseteq EG'$ a *(T',d)-join* if (i) $\deg_{B'}(v)\equiv d_v \pmod 2$ for each $v\in VG'$, and (ii) the graph (VG',B') contains a set of mutually edge-disjoint T'-paths such that for each $s'\in T'$, exactly $d_{s'}$ of these paths begin or end at s'; here $\deg_{B'}(v)$ is the number of edges in B' incident to a node v, and we let $d_v=0$ for $v\in VG'-T'$. The set of (T',d)-joins in G' is denoted by $\mathcal{B}_d$. Consider the *minimum weight (T',d)-join problem*:

(1.14) *given $w:EG'\to\mathbb{Z}_+$, find a (T',d)-join B' of weight $w(B')$ minimum.*

If $d=\mathbb{1}$ then $|T'|$ is even and we get the well-known notion of *T'-join* (up to the usual requirement of being inclusion minimal); such an object arises in connection with the Chinese postman problem [16,7]. Edmonds and Johnson [9] proved that the minimum weight of a T'-join is equal to the maximum value of a (fractional) w-packing of T'-cuts ($\delta^{G'}(X)$ is called a *T'-cut* if $|X\cap T'|$ is odd). In polyhedral terms, this means that the *dominant*

$$\mathrm{dom}(\mathcal{B}_{\mathbf{1}}) = \mathrm{conv}(\mathcal{B}_{\mathbf{1}}) + \mathbb{R}_+^{EG'}$$

for $\mathcal{B}_{\mathbf{1}}$ is formed by the vectors $x' \in \mathbb{R}_+^{EG'}$ satisfying $x'(\delta(X)) \geq 1$ for all T'-cuts $\delta(X)$. [For a family $\mathcal{L}$ of subsets of a set E, $\mathrm{conv}(\mathcal{L})$ is the convex hull of the incidence vectors $\xi_L \in \mathbb{R}^E$ of sets $L \in \mathcal{L}$, and for sets $\mathcal{X}, \mathcal{Y} \subseteq \mathbb{R}^E$, $\mathcal{X}+\mathcal{Y}$ denotes their Minkowsky sum $\{z\ :\ z = x+y$ some $x \in \mathcal{X}$ and $y \in \mathcal{Y}\}$.] Also there are other "nice properties" of T'-joins and T'-cuts and strongly polynomial algorithms to solve (1.14) with $d = 1\!\!1$. For a survey, see, e.g. [15,30].

In case of arbitrary $d \in \mathbb{Z}_+^{T'}$, problem (1.14) becomes more involved. We reduce it to (1.3) with $H = K_T$ and then apply results on the latter. More precisely, for each $s' \in T'$, add to G' a new node s and $d_{s'}$ parallel edges between s and s'. The resulting graph G is called the *d-extension* of G'. Let $T = \{s\ :\ s' \in T'\}$ and $c_e = 1$ for all $e \in EG$. Assign the cost a_e to be w_e if $e \in EG'$, and 0 if $e \in EG - EG'$. Clearly, $\mathcal{B}_d \neq 0$ if and only if $2\nu(G, K_T, 1\!\!1) = d(T')$, in which case the algorithm for (1.7) yields an optimal solution to (1.14). Moreover, Theorem 3 enables us to describe the dominant for $\mathcal{B}_d$ as follows.

A pair $\phi = (X_\phi, U_\phi)$ is called a *fragment* for G', T', d if $X_\phi \subseteq VG'$, $U_\phi \subseteq \delta^{G'}(X_\phi)$, and the numbers $|U_\phi|$ and $d(X_\phi \cap T')$ have different parity, that is,

$$|U_\phi| - \sum(d_{s'}\ :\ s' \in X_\phi \cap T') \equiv 1 \pmod 2. \tag{1.15}$$

In particular, if $X_\phi \cap T' = \emptyset$, it turns into the above definition of inner fragments. The characteristic function of a fragment is defined as in (1.8) (concerning G').

Theorem 5 [3]. *Let D' be the set of vectors $x' \in \mathbb{R}^{EG'}$ satisfying*

(1.16) (i) $\quad 0 \leq x'_e \leq 1 \quad$ *for* $e \in EG'$;

(ii) $\quad x'(\delta(Y)) \geq d_{s'} - d(Y \cap T' - \{s'\})$ *for* $Y \subseteq VG'$ *and* $s' \in Y \cap T'$;

(iii) $\quad x'\chi_\phi \leq |U_\phi| - 1 \quad$ *for each fragment* ϕ.

Then $\mathrm{conv}(\mathcal{B}_d) \subseteq D' \subseteq \mathrm{dom}(\mathcal{B}_d)$. *In particular,* $\mathrm{dom}(\mathcal{B}_d) = D' + \mathbb{R}_+^{EG'}$.

Note that for $d = 1\!\!1$ the polyhedron D' is exactly $\mathrm{conv}(\mathcal{B}_{\mathbf{1}})$. Indeed, $\mathrm{conv}(\mathcal{B}_{\mathbf{1}})$ is described by (1.16)(i) together with the following constraints (see [15], Ch. 8.5): for each $X \subseteq VG'$ and $W \subseteq \delta(X)$ with $|X \cap T'| + |W|$ odd,

$$x'(\delta(X) - W) + |W| - x'(W) \geq 1.$$

The latter constraints are equivalent to (1.16)(iii) (for $d = 1\!\!1$), whence $D' \subseteq \mathrm{conv}(\mathcal{B}_{\mathbf{1}})$ (and therefore, these polyhedra coincide). In general case D' needs not coincide with $\mathrm{conv}(\mathcal{B}_d)$, as mentioned in Section 6.

We discuss Theorem 5 in Section 5. An analogous description exists for the dominant of the set $\mathcal{B}^{\max}$ of maximum multi-joins for G', T'. Here by a *maximum multi-join* we mean a subset $B' \subseteq EG'$ such that $\deg_{B'}(v)$ is even for each $v \in VG' - T'$ and the subgraph (VG', B') contains $\nu = \nu(G', K_{T'}, 1\!\!1)$ pairwise edge-disjoint T'-paths.

A collection $K = \{Y_{s'} \,:\, s' \in T'\}$ of pairwise disjoint sets $Y_{s'} \subset VG'$ with $Y_{s'} \cap T' = \{s'\}$ is called a *T'-kernel family*. For $e \in EG'$ define $\zeta_K(e)$ to be the number of occurrencies of e in the cuts $\delta(Y_{s'})$ among $s' \in T$ (thus $\zeta_K(e)$ is 0, 1 or 2).

Theorem 6 [3]. *Let Q' be the set of vectors $x' \in \mathbb{R}^{EG'}$ satisfying*

$$
\text{(1.17)} \qquad \begin{array}{lll}
\text{(i)} & 0 \le x'_e \le 1 & \textit{for } e \in EG'; \\
\text{(ii)} & x'\zeta_K \ge 2\nu & \textit{for each } T'\textit{-kernel family } K; \\
\text{(iii)} & x'\chi_\phi \le |U_\phi| - 1 & \textit{for each inner fragment } \phi.
\end{array}
$$

Then $\operatorname{conv}(\mathcal{B}^{\max}) \subseteq Q' \subseteq \operatorname{dom}(\mathcal{B}^{\max})$. *In particular,* $\operatorname{dom}(\mathcal{B}^{\max}) = Q' + \mathbb{R}_+^{EG'}$.

Finally, in concluding Section 6 we discuss an analog of (1.7) to openly disjoint T-paths and raise open questions.

2. Sketch of the proof of Theorem 1.

For details, we refer the reader to [21]. Let f and γ be o.s. to (1.5) and (1.6), respectively. We show the existence of a half-integer o.s. f' to (1.5). We may assume that $a_e > 0$ for all $e \in EG$ (as validity of the theorem for all *positive* a's implies that for all *nonnegative* a's, by obvious perturbation arguments).

Define the *length* function ℓ on EG to be $a + \gamma$; then ℓ is positive. Applying the linear programming duality theorem to (1.5)-(1.6), we observe that feasible f and γ are optimal if and only if they satisfy the (complementary slackness) conditions:

(2.1) $f_P > 0$ implies $\ell(P) = p$; in particular, P is an ℓ-shortest path;

(2.2) $\gamma_e > 0$ implies $\zeta^f(e) = c_e$, i.e., e is *saturated* by f.

We may assume that $p > 0$ and $\min\{\mathrm{dist}_\ell(s,t) \,:\, s,t \in T,\ s \ne t\} = p$ (otherwise $f = \mathbb{O}$, by (2.1), and we are done). Let $\mathcal{P}^p = \mathcal{P}^p(\ell)$ be the set of T-paths of ℓ-length exactly p. Extract the subgraph $G^p = G^p(\ell)$ of G that is the union of T and all paths in $\mathcal{P}^p$. Let $\mathrm{dist}(\cdot,\cdot)$ stand for $\mathrm{dist}_\ell(\cdot,\cdot)$.

Consider $v \in VG^p$. The *potential* $\pi(v)$ of v is the distance from v to T, i.e., $\min\{\mathrm{dist}(v,s) \,:\, s \in T\}$. Denote by $T(v)$ the set of terminals $s \in T$ with $\mathrm{dist}(v,s) = \pi(v)$. Clearly $\pi(v) \le p/2$. Moreover, if $\pi(v) < p/2$ then $|T(v)| = 1$, while if $\pi(v) = p/2$ then $|T(v)| \ge 2$. Thus, VG^p is partitioned into the sets $V^\bullet = \{v \in VG^p \,:\, |T(v)| \ge 2\}$ and $V_s = \{v \in VG^p \,:\, T(v) = \{s\}\}$, $s \in T$. The positivity of ℓ provides the following properties.

(2.3) Let $e = uv \in EG$ and $u, v \in VG^p$. Then $e \in EG^p$ if and only if, up to permutation of u and v, either (i) $u \in V_s$, $v \in V_s \cup V^\bullet$ and $\pi(v) - \pi(u) = \ell_e$ for some $s \in T$, or (ii) $u \in V_s$, $v \in V_t$ and $\pi(u) + \pi(v) + \ell_e = p$ for some distinct $s, t \in T$.

(2.4) Let $P = (v_0, e_1, v_1, \ldots, e_k, v_k)$ be a path in G^p connecting distinct terminals $s = v_0$ and $t = v_k$. Then $P \in \mathcal{P}^p$ if and only if there is $0 \le i < k$ such that $v_0, \ldots, v_i \in V_s$; $v_{i+2}, \ldots, v_k \in V_t$; $\pi(v_0) < \ldots < \pi(v_i)$; $\pi(v_{i+2}) > \ldots > \pi(v_k)$; and either $v_{i+1} \in V^\bullet$, or $v_{i+1} \in V_t$ and $\pi(v_{i+1}) > \pi(v_{i+2})$.

Property (2.3) enables us to construct a digraph $\Gamma = (V\Gamma, A\Gamma)$, the *double covering* over G^p, as follows. Split each $v \in VG^p$ into $2|T(v)|$ nodes v_s^1 and v_s^2 $(s \in T(v))$. If $T(v) = \{s\}$, we also denote v_s^i as v^i. The arcs of Γ are assigned as follows:

(i) each $e = uv \in EG^p$ with $u \in V_s$, $v \in V_s \cup V^\bullet$ and $\pi(u) < \pi(v)$ generates two arcs (u_s^1, v_s^1) and (v_s^2, u_s^2);

(ii) each $e = uv \in EG^p$ with $u \in V_s$ and $v \in V_t$ $(s \ne t)$ generates two arcs (u_s^1, v_t^2) and (v_t^1, u_s^2);

(iii) each $v \in V^\bullet$ generates arcs (v_s^1, v_t^2) for all distinct $s, t \in T(v)$;

see Fig. 3 where $T = \{s, t, q\}$, $p = 4$, the numbers on edges indicate values of ℓ, and the arcs of Γ are directed up. Arcs in (i) and (ii) have capacities c_e, and arcs in (iii) have capacity ∞. We use the same notation c for the capacities in Γ and think of $T^1 = \{s^1 : s \in T\}$ and $T^2 = \{s^2 : s \in T\}$ as the sets of *sources* and *sinks* of Γ, respectively. Define $\sigma(v_s^i) = v_s^{3-i}$. This gives a *skew symmetry* of Γ because for each $b = (u_s^i, v_t^j) \in A\Gamma$, (v_t^{3-j}, u_s^{3-i}) is also an arc of Γ, denoted as $\sigma(b)$. We extend σ to the dipaths of Γ in a natural way.

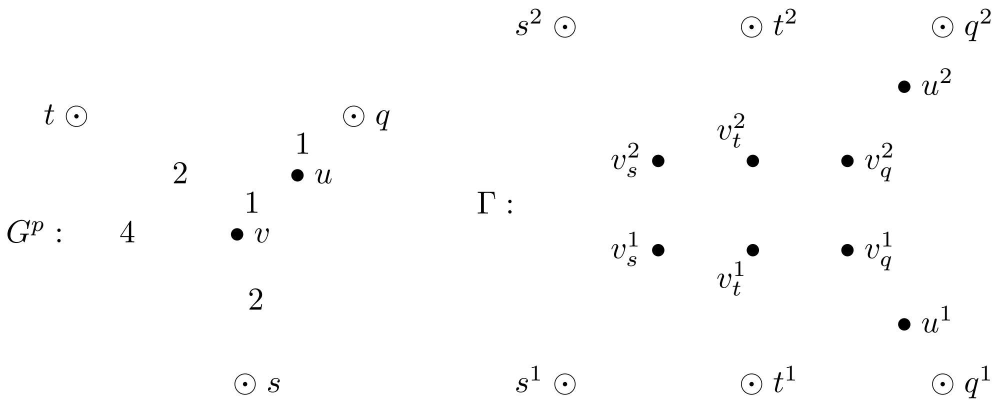

Fig. 3

The construction of Γ yields a natural mapping ω of $V\Gamma \cup A\Gamma$ to $VG^p \cup EG^p$; it brings a node v_s^i to v, an arc (y_s^i, z_t^j) as in (i) or (ii) to the edge yz, and an arc (v_s^1, v_t^2) as in (iii) to the node v. We extend ω in a natural way to a mapping of the dipaths of Γ into paths of G^p. From (2.4) one can derive the following key property:

(2.5) (i) for a dipath P in Γ, P and $\sigma(P)$ are disjoint, and $\omega(\sigma(P))$ is reverse to $\omega(P)$;

(ii) ω yields a one-to-one correspondence between $\mathcal{P}^p$ and the set of T^1 to T^2 dipaths in Γ.

Such a correspondence is further extended to a relationship between multiflows on $\mathcal{P}^p$ and flows in Γ, as follows. By a *flow* in Γ we mean a function $h : A\Gamma \to \mathbb{Q}_+$ satisfying the conservation condition

$$\mathrm{div}_h(y) := \sum_{z:(y,z)\in A\Gamma} h_{(y,z)} - \sum_{z:(z,y)\in A\Gamma} h_{(z,y)} = 0 \quad \text{for all} \;\; y \in V\Gamma - (T^1 \cup T^2),$$

and the capacity constraint $h_b \le c_b$ for all $b \in A\Gamma$. Obviously, a flow h can be represented as the sum of elementary flows along dipaths (note that Γ is acyclic). That is, there are T^1 to T^2 dipaths $P_1, \ldots, P_m$ and rationals $\alpha_1, \ldots, \alpha_m \ge 0$ such that $\sum(\alpha_i \,:\, b \in P_i) = h_b$ for $b \in A\Gamma$; we call $\mathcal{D} = \{(P_i, \alpha_i)\}$ a *decomposition* of h. Such a $\mathcal{D}$ induces the function $g = g^{\mathcal{D}}$ on $\mathcal{P}^p$ by setting $g_{\omega(P_i)} = \alpha_i/2$ for $i = 1, \ldots, m$ and $g_P = 0$ for the remaining members P of $\mathcal{P}^p$. We observe that for any $b \in A\Gamma$ with $e = \omega(b) \in EG^p$,

$$\zeta^g(e) = \frac{1}{2}(h_b + h_{\sigma(b)}) \le \frac{1}{2}(c_b + c_{\sigma(b)}) = c_e, \tag{2.6}$$

hence, g gives a c-admissible multiflow. Conversely, let $g : \mathcal{P}^p \to \mathbb{Q}_+$ be c-admissible. Define the function $h = h^g$ on $A\Gamma$ so that for $b \in A\Gamma$, h_b is the sum of values $g_{\omega(P)}$ over all T^1 to T^2 dipaths P in Γ that contain b or $\sigma(b)$. Then h is a flow, and for $b \in A\Gamma$ with $e = \omega(b) \in EG^p$ we have

$$h_b = h_{\sigma(b)} = \zeta^g(e). \tag{2.7}$$

Now we are able to prove Theorem 1. Given f as above, form the flow $h = h^f$ in Γ. Let $E^+ = \{e \in EG^p \,:\, \gamma_e > 0\}$. Then each $e \in E^+$ is saturated by f (by (2.2)), whence $h_b = c_b$ for $b \in A^+ := \omega^{-1}(E^+)$ (by (2.7)). Since all capacities in Γ are integers, there exists an *integer* flow h' with $h'_b = h_b = c_b$ for all $b \in A^+$. Choose a decomposition $\mathcal{D} = \{(P_i, \alpha_i)\}$ of h' with all α_i's integral. Let $f' = g^{\mathcal{D}}$. By (2.6), f' is a c-admissibbe function on $\mathcal{P}^p$. Moreover, f' is half-integral and it saturates all edges in E^+. Thus, f' and γ satisfy (2.1) and (2.2), so f' is the desired o.s. to (1.5).

In conclusion of this section we explain how to find a half-integer o.s. to (1.5) (and therefore, to (1.2) with $H = K_T$) in strongly polynomial time. Again, we may assume that a is positive. For if $Z = \{e \,:\, a_e = 0\} \ne \emptyset$, we can replace a by a' defined by $a'_e = 1$ for $e \in Z$ and $a'_e = (2c(Z) + 1)a_e$ otherwise, using the fact that there are half-integer o.s. for a and for a'. First, find an o.s. γ to (1.6) by use of the version [35] of the ellipsoid method [25]; it takes time polynomial in $n = |VG|$ since the size of the constraint matrix behind (1.6) is polynomial in n. Second, form G^p for given p and $\ell = a + \gamma$ (by solving corresponding shortest paths problems) and construct Γ over

G^p. Third, find an integer flow h in Γ with the restrictions $h_b = c_b$ for $b \in A^+$ (such an h must exist). Now an integer decomposition of h determines the desired half-integer multiflow for G.

3. Unbounded fractionality

As mentioned in the Introduction, to prove Theorem 2 it suffices to show that $\varphi(H) = \infty$ for $H = H_1, H_2, H_3$ as in Fig 1. Following [20], we design "bad networks" $N = (G, H, c, a)$ for these H's. Let k be an odd positive integer. Take k disjoint paths $(v_1^i, e_2^i, v_2^i, \dots, e_{2k}^i, v_{2k}^i)$, $i = 1, \dots, k$. Connect v_j^i and v_j^{i+1} by edge u_j^i for all i, j such that $i - j \equiv 1 \pmod 2$. Add nodes $s, t, s', t', y, z, y', z'$ and edges

(i) $sy, tz, s'y', t'z'$;

(ii) yv_1^i and zv_{2k}^i for $i = 1, \dots, k$;

(iii) $y'v_j^1$ for each odd j, and $z'v_j^k$ for each even j,

obtaining graph G. Assign the capacity $k-1$ to the edges $s'y', t'z'$, and 1 to the other edges of G. Assign the edge costs as follows:

0	for tz and e_{2j}^i, $i, j = 1, \dots, k$;
1	for all edges u_j^i and the remaining edges e_j^i;
k	for $s'y', t'z'$ and the edges as in (ii) and (iii);
$2k$	for sy.

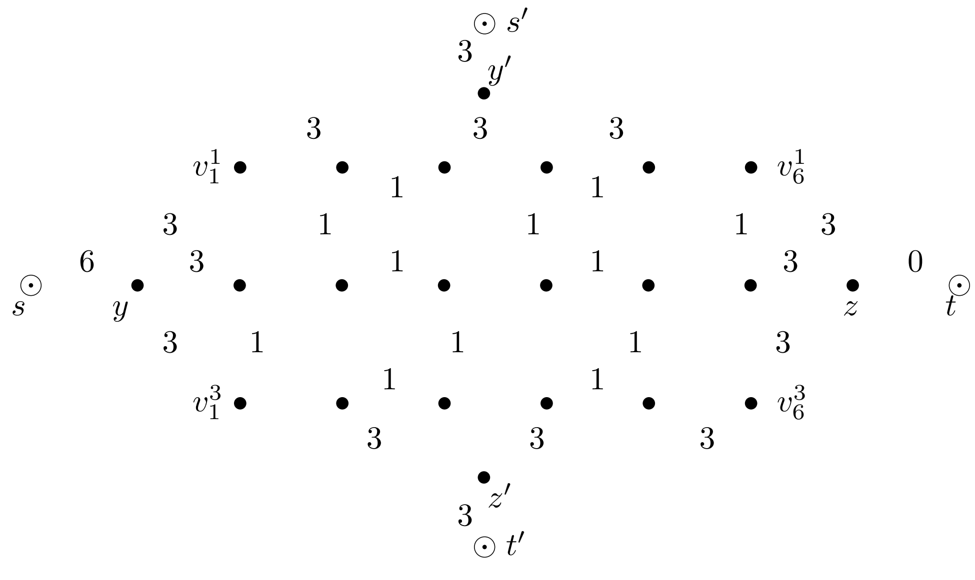

Fig. 4

(See Fig. 4 where $k = 3$ and the numbers on edges indicate non-zero costs.) We identify s, t, s', t' with the corresponding nodes of the graph $H \in \{H_1, H_2, H_3\}$ in question; therefore $\{st, s't'\} \subseteq EH \subseteq \{st, s't', ss', st'\}$.

For $i = 1, \dots, k$, let P_i (L_i) be the simple path going through the nodes $s, y, v_1^i, \dots, v_{2k}^i, z, t$ (respectively, $s', y', v_{2i-1}^1, v_{2i}^1, v_{2i}^2, v_{2i-1}^2, \dots, v_{2i-1}^{k-1}, v_{2i-1}^k, v_{2i}^k, z', t'$). Assign multiflow f by $f_{P_i} = 1/k$, $f_{L_i} = (k-1)/k$ $(i = 1, \dots, k)$ and $f_Q = 0$ for the other paths Q in $\mathcal{P}(G, H)$. One can check that:

(i) f satisfies the capacity constraints and saturates $sy, tz, s'y', t'z'$, whence f is a maximum multiflow in N;

(ii) the cost of any path in $\mathcal{P}(G, H)$ is at least $5k-1$, and it equals $5k-1$ for the only paths P_i's and L_i, $i = 1, \dots, k$;

(iii) f is an only maximum multiflow with the support in $\{P_1, \dots, P_k, L_1, \dots, L_k\}$.

Thus, problem (1.2) for our network has a unique o.s. f, and this f has components with denominator k. Since k can be chosen arbitrarily large, $\varphi(H) = \infty$.

4. Sketch of the proof of Theorem 3

We outline main ideas of the proof, not coming into particular details; for the complete proof, see [22]. As before, we may assume that a is positive.

Given p, we say that a packing $\mathcal{D}$ and a p-admissible (β, γ) are *good* if they achieve the equality in (1.12). Obviously, $\mathcal{D} = \emptyset$ and $(\beta, \gamma) = (\mathbb{0}, \mathbb{0})$ are good for $p = 0$. We grow p from 0 through ∞ and show the existence of good $\mathcal{D}, \beta, \gamma$ for every value of p. More precisely, Theorem 3 is easily reduced to the following theorem.

Theorem 4.1. *Let $\mathcal{D}, \beta, \gamma$ be good for some p. Then one of the following is true:*

(i) *there exists a packing $\mathcal{D}'$ with $|\mathcal{D}'| = |\mathcal{D}| + 1$ such that $\mathcal{D}', \beta, \gamma$ are good for p;*

(ii) *there exist $p' > p$ and (β', γ') such that for any $0 \le \xi \le 1$, the packing $\mathcal{D}$ and functions $(1-\xi)\beta + \xi\beta'$ and $(1-\xi)\gamma + \xi\gamma'$ are good for $(1-\xi)p + \xi p'$.*

The key idea in the proof of Theorem 4.1 is that in place of packings we can handle certain subsets of edges of G. Let $\ell = \ell^{\beta,\gamma}$ be as in (1.9).

Definition. Given a p-admissible (β, γ), a subset $B \subseteq EG$ is called *regular* if $B = \cup(P \in \mathcal{D})$, where $\mathcal{D}$ is a packing consisting of simple T-paths P with $\ell(P) = p$.

For $E' \subseteq EG$ and $v \in VG$ let $E'(v)$ denote the set of edges in E' incident to v. The *value* $\mathrm{val}(B)$ of a regular B is defined as $\frac{1}{2}\sum(|B(s)| : s \in T)$; clearly $\mathrm{val}(B)$ equals the cardinality of a packing $\mathcal{D}$ as above, unless $p = 0$. We say that B, β, γ are good (for p) if $\mathcal{D}, \beta, \gamma$ are good. Considering the inequalities in (1.13), we observe that B, β, γ are good if and only if:

(4.1) $\quad \gamma_e > 0$ implies $e \in B$;

(4.2) $\quad \beta_\phi > 0$ implies $\chi_\phi(B) = |U_\phi| - 1$ (ϕ is *saturated* by B).

Let $\overline{U}_\phi$ denote $\delta(X_\phi) - U_\phi$. The equality in (4.2) is possible in two cases, namely,

(4.3) $\quad$ either (i) B contains all but one element of U_ϕ and none of $\overline{U}_\phi$, or (ii) B includes U_ϕ and meets exactly one element of $\overline{U}_\phi$.

This only element of $U_\phi - B$ in case (i), and of $B - U_\phi$ in case (ii) is called the *root* of ϕ and denoted by r_ϕ. In the proof of Theorem 4.1 we impose some additional conditions (inductive assumptions) on B, β, γ we deal with. Let $\widehat{\mathcal{F}} = \{\phi \in \mathcal{F}^0 : \beta_\phi > 0\}$, and let G^p be the subgraph of G that is the union of T and all (not necessarily simple) T-paths of ℓ-length p. The first group consists of four conditions:

(A0) for $\phi \in \widehat{\mathcal{F}}$, r_ϕ is in G^p and $\gamma_{r_\phi} = 0$;

(A1) for distinct $\phi, \phi' \in \widehat{\mathcal{F}}$, either $X_\phi \cap X_{\phi'} = \emptyset$ or $X_\phi \subset X_{\phi'}$ or $X_{\phi'} \subset X_\phi$;

(A2) for $\phi_1, \phi_2 \in \widehat{\mathcal{F}}$ with $X_{\phi_1} \subset X_{\phi_2}$ and $i = 1, 2$, the set U_{ϕ_i} includes $U_{\phi_{3-i}} \cap \delta(X_{\phi_i})$ (or, equivalently, if $r_{\phi_{3-i}} \in \delta(X_{\phi_i})$ then $r_{\phi_1} = r_{\phi_2}$);

(A3) there are no $\phi_1, \ldots, \phi_k \in \widehat{\mathcal{F}}$ with $k > 1$ such that the sets X_{ϕ_i} are pairwise disjoint, and $r_{\phi_i} \in \delta(X_{\phi_{i+1}})$, $i = 1, \ldots, k$ (letting $\phi_{k+1} = \phi_1$).

The regularity of B implies $B \subseteq EG^p$. Let $J = \{e \in EG : \ell_e = 0\}$ (although a is positive, $\ell_e = 0$ is possible since χ_ϕ takes negative values on $\overline{U}_\phi$). Define $B^0 = B \cap J$ and $B^+ = B - J$. The second group of conditions concerns J, namely,

(A4) each $e = uv \in J$ belongs to G^p, $\gamma_e = 0$, and none of u, v is in T;

(A5) for each $\phi \in \widehat{\mathcal{F}}$, $J \cap U_\phi = \emptyset$; in particular, $e \in B^0 \cap \delta(X_\phi)$ implies $e = r_\phi$.

A component of the graph $(VG^p - T, B^0)$ is called a 0-*component* (by (A4), B^0 does not meet T); a 0-component Q is called *trivial* if $EQ = \emptyset$. We observe that each 0-component is a tree. Indeed, for a circuit C in (VG, B) and $\phi \in \mathcal{F}^0$, we have $|U_\phi \cap C| \geq |\overline{U}_\phi \cap C|$ (because of (4.3)), whence $\chi_\phi(C) \geq 0$. Therefore, $\ell(C) - a(C) - \gamma(C) = \sum(\beta_\phi \chi_\phi(C) : \phi \in \mathcal{F}^0) \geq 0$. Since $\gamma(C) \geq 0$ and $a(C) > 0$, $\ell(C)$ is nonzero. In what follows we refer to a connected subgraph of $(VG^p - T, B^0)$ as a 0-*tree*.

The current B is transformed into a new regular set B' of bigger value by use of a certain *augmenting path*. Construction of such a path appeals to two notions. The first notion concerns so-called attachments. To introduce them, we identify T with the set of integers from 1 through $|T|$ and let $\langle T \rangle$ be the set $\{-|T|, \ldots, -1, 1, \ldots, |T|\}$. Define $Z = EG^p - B$ and $Z^0 = Z \cap J$.

For the nodes in G^p define the potentials π and sets $V^\bullet$ and V_s ($s \in T$) as in Section 2 with respect to our ℓ. For $v \in VG^p$ and $e = uv \in EG^p$ with $\ell_e > 0$, we assign the *attachment* $\alpha(v, e) \in \langle T \rangle$ by the following rule:

(4.4) (i) if $v \in V_s \cup V^\bullet$, $u \in V_s$ and $\pi(u) < \pi(v)$, set $\alpha(v, e) = s$;

(ii) if $v \in V_s$ and either $u \notin V_s$, or $u \in V_s$ and $\pi(u) > \pi(v)$, set $\alpha(v, e) = -s$.

If $e = uv \in Z^0$, (v, e) is provided with the special attachment $\alpha(v, e) = 0$. The attachments for edges in B^0 are assigned in a more sophisticated way. Note that both ends of an edge in J (and therefore, the nodes of a 0-component) have the same potentials and belong to the same set among the V_s's and $V^\bullet$. For a subgraph Q of

G^p let $B(Q)$ ($B^+(Q)$) denote the set of edges in B (resp. B^+) with exactly one end in Q. For $s \in \langle T\rangle$ define $B^+_s(Q)$ to be the set of edges $e = uv \in B^+(Q)$ with $v \in VQ$ and $\alpha(v, e) = s$. The next lemma follows from (2.4) if $B^0 = \emptyset$; in general case, the part "only if" is easy, while the part "if" is proved by induction on $|B|$. Here $E' \subseteq EG$ is called *inner Eulerian* if $|E'(v)|$ is even for each $v \in VG - T$.

Lemma 4.2. *$B \subseteq EG^p$ is regular if and only if B is inner Eulerian and*

(4.5) $\quad |B^+_s(Q)| \le |B(Q)|/2$ *for each $s \in \langle T\rangle$ and 0-tree Q for p, ℓ, B.*

If the inequality in (4.5) holds with equality, we say that s is *tight* for Q. For example, if $Q = (\{v\}, \emptyset)$ is a trivial 0-component with $v \in V_s$, then $\{B^+_s(v), B^+_{-s}(v)\}$ gives a partition of $B(v)$, and both s and $-s$ are tight for Q. One can see that:

(4.6) (i) if s is tight for a 0-tree Q, then for any $e \in EQ'$, s is tight for exactly one of two components of $Q' - \{e\}$;

(ii) for $e = uv \in B^0$, there is at most one $s \in \langle T\rangle$ such that s is tight for some 0-tree that contains u but v.

Based on (4.6)(ii), we assign the attachment to each edge $e = uv \in B^0$ as follows:

(4.7) set $\alpha(v, e) = s$ if $s \in \langle T\rangle$ is tight for some 0-tree Q with $v \notin VQ \ni u$, and $\alpha(v, e) = 0$ otherwise.

For $v \in VG^p$ let $E(v)$ stand for $EG^p(v)$. For $s \in \langle T\rangle \cup \{0\}$ define

$$E_s(v) = \{e \in E(v) : \alpha(v, e) = s\}, \quad B_s(v) = B \cap E_s(v) \quad \text{ and } \ Z_s(v) = Z \cap E_s(v).$$

Using (2.4) and (4.6)(i), one can check that the resulting attachments satisfy:

(4.8) for $e = uv \in VG^p$, $\alpha(v, e) \ne \alpha(u, e)$ unless $\alpha(v, e) = \alpha(u, e) = 0$;

(4.9) $|B_s(v)| \le |B(v)|/2$ for any $v \in VG^p - T$ and $s \in \langle T\rangle$.

The second notion needed to define an augmenting path concerns so-called forks. Let $\mathcal{F}^{\max}$ be the set of $\phi \in \widehat{\mathcal{F}}$ with X_ϕ maximal; by (A1), the sets X_ϕ, $\phi \in \mathcal{F}^{\max}$, are pairwise disjoint. For each $\phi \in \mathcal{F}^{\max}$ shrink in G^p the subgraph $\langle X_\phi\rangle_{G^p}$ induced by X_ϕ into node v_ϕ, forming graph G^*. These v_ϕ's are called the *fragment-nodes*, whereas the other (non-shrunk) nodes in G^* are called *ordinary*; we keep the same notation for corresponding edges in G^p and G^*. Consider a node $v \in VG^* - T$ and distinct edges e, e' in G^* incident to v. We say that $\tau = (v, e, e')$ is a *fork* if

(4.10) (i) v is ordinary and there is no $s \in \langle T\rangle$ tight for v with $e, e' \in Z_s(v) \cup (B(v) - B_s(v))$; or

(ii) v is a fragment-node v_ϕ and one of e, e' is r_ϕ.

In other words, (4.10)(i) means that the transformation $B \to B \triangle \{e, e'\}$ preserves the regularity (i.e., (4.9)) at v; here $\triangle$ denotes the symmetric difference. Similarly to making G^*, for each $\phi \in \widehat{\mathcal{F}}$, we form graph G_ϕ from $\langle X_\phi \rangle_{G^p} \cup \{r_\phi\}$ by shrinking $\langle X_{\phi'} \rangle_{G^p}$ into node $v_{\phi'}$ for each $\phi' \in \mathcal{F}_\phi$, where $\mathcal{F}_\phi$ denotes the set of $\phi' \in \widehat{\mathcal{F}}$ with $X_{\phi'}$ maximal provided that $X_{\phi'} \subset X_\phi$. Let X^*_ϕ denote the image of X_ϕ in G_ϕ. We define the forks in G_ϕ as in (4.10) (concerning triples in G_ϕ).

Next, it is convenient to deal with augmenting paths which are non-self-intersecting on edges. To this purpose, one trick is applied. We slightly modify G^p by adding, for each $e \in J$, a parallel edge e', the *mate* of e, and consider e' as an element of Z with $\ell_{e'} = 0$. Accordingly, we correct G^* and the G_ϕ's. This slightly extends the set of forks in (4.10); e.g., for $e = uv \in J$ and its mate e', (v, e, e') is always a fork since $\alpha(v, e') = 0$. We keep notations $E(v), Z(v), E_s(v)$ and etc. for the corresponding sets in the new G^p.

An edge $e \in EG^p$ with $\gamma_e = 0$ is called *feasible*. A path $P = (v_0, e_1, v_1, \ldots, e_k, v_k)$ in G^* is called *active* if: (i) $e_1, \ldots, e_k$ are distinct and feasible; (ii) $v_0 \in T$, $e_1 \in Z$ and $v_1, \ldots, v_{k-1} \notin T$; and (iii) (v_i, e_i, e_{i+1}) is a fork, $i = 1, \ldots, k-1$. If, in addition, (v_i, e_i, e_{j+1}) is not a fork for any $1 \le i < j < k$ with $v_i = v_j$, P is called *minimal* active. If, further, $v_k \in T$ and $e_k \in Z$, P is called *augmenting* in G^*. An active path in G_ϕ is defined by replacing (ii) by (ii') $e_1 = r_\phi$ and $v_1, \ldots, v_k \in X^*_\phi$. One can show (using (4.10) and (4.14) below) that a minimal active path can pass through each ordinary node at most twice and through each fragment-node at most once.

An augmenting path P' in G^p is constructed from an augmenting path P in G^* as follows. If all nodes in P are ordinary then P is already the desired P'. Otherwise we repeatedly apply the replacement procedure described below, assuming that each $\phi \in \widehat{\mathcal{F}}$ satisfies the following "strong reachability" condition:

(A6) (i) each fragment-node $v_{\phi'} \in X^*_\phi$ is reachable in G_ϕ by an active path with the last edge $r_{\phi'}$;

(ii) each ordinary node $v \in X^*_\phi$ is reachable in G_ϕ by an active path L_v; moreover, if $s \in \langle T \rangle$ is tight for v, then there is a minimal active path L^s_v to v such that $|B''_s(v)| = (|B''(v)| - 1)/2$, where $B'' = B \triangle L^s_v$ and B''_s is defined with respect to the attacments for B.

We also assume that the paths required in (A6) can be efficiently found. In the *replacement procedure*, we choose in P a fragment-node v_ϕ and examine the edges e, e' of P adjacent to v_ϕ. By (4.10)(ii), one of these, e say, is r_ϕ. Let w be the end of e' in X^*_ϕ. We replace v_ϕ by a minimal active path L (without r_ϕ) in G_ϕ coming to w. If $w = v_{\phi'}$ for some $\phi' \in \mathcal{F}_\phi$, we take as L the path as in (A6)(i). If w is ordinary, we choose as L the path L_w or L^s_w for some s as in (A6)(ii). The rule of choosing s in the latter case depends on the 0-component Q containing w and the edges of P incident to nodes in Q, and we omit it here. The procedure is repeated for the new P (and a maximal fragment in $\widehat{\mathcal{F}}$ with ϕ discarded), and so on until no fragment-node in the current path exists. The resulting path is just the desired augmenting path in G^p.

Once an augmenting path P' in G^p is found, we transform B into $B' = B \triangle P'$. Note that if P' contains the mate e' of an original edge e but e itself, then (A5) implies that $e \in Z$, so we can exchange e' and e. And if P' contains both an original edge e and its mate e', then e must be the root of some fragment ϕ in $\widehat{\mathcal{F}}$, whence $e \in B$; in this case e remains in B'.

We observe that, regardless of the choice of active paths as in (A6)(ii), if a cut $\delta(X_\phi)$ for some $\phi \in \widehat{\mathcal{F}}$ meets P', then P' intersects it twice, passing r_ϕ and some other edge e. This implies that ϕ remains saturated ($\chi_\phi(B') = |U_\phi| - 1$), and e becomes the new root of ϕ (unless e is the mate of r_ϕ). Thus, (4.2) remains true. It is easy to see that (4.1) and (A0)-(A5) continue to hold. Furthermore, B' is inner Eulerian, and $\mathrm{val}(B') = \mathrm{val}(B) + 1$. The following key lemma shows the regularity of B'.

Lemma 4.3. *(4.5) holds for B'.*

In the simplest case, when Q is a trivial 0-tree $(\{v\}, \emptyset)$ with $v \in VG^*$ and $\ell_e > 0$ for all $e \in B(v) \cup B'(v)$, this easily follows from the definitions of attachments and forks (cf. (4.4) and (4.10)(i)). In other cases, the proof is more complicated; it applies induction on the number of replacements and uses (4.6)(ii) and the property (provided by (A5)) that if a 0-component Q' for B meets X^*_ϕ for some $\phi \in \widehat{\mathcal{F}}$, then either Q' entirely lies in $\langle X_\phi \rangle_{G^p}$ or r_ϕ is an only common edge in Q' and $\delta(X_\phi)$.

Thus, the existence of an augmenting path as above leads to alternative (i) in Theorem 4.1. To be consistent, one also proves that (A6) preserves for B' and the corresponding roots and attachments.

Now we explain how to find an augmenting path in G^* efficiently and then show that if such a path does not exist, then alternative (ii) in Theorem 4.1 takes place. We apply a *labelling method* similar, in a sense, to that used for finding alternating paths in matching problems. This grows a digraph whose underlying graph uses feasible edges of G^*. Equivalently, a feasible edge $e = uv \in EG^*$ can be in one of three possible current states: e is either *unlabelled*, or *labelled in one direction*, from u to v say, or *labelled in both directions*, from u to v and reversely; let $E^{\mathrm{un}}, E^1, E^2$ denote the sets of such edges, respectively.

The labelling process is organized so that for each edge $e = uv$ labelled from u to v, there is an active path containing u, e, v in this order and with all edges already labelled at least in forward direction. It terminates when (i) some edge $e = uv \in Z$ with $v \in T$ becomes labelled from u to v, or (ii) one can no longer label edges so that E^1 or E^2 increases. In case (i), we get an augmenting path.

Assume that the process terminates with case (ii) (and not (i)). A node $v \in VG^* - T$ is called 1-*labelled* if it is incident to an edge in E^1 but none of E^2; for such a v denote by $E^{\mathrm{in}}(v)$ ($E^{\mathrm{out}}(v)$) the set of edges in $E^1(v)$ labelled to (respectively, from) v. A component of the subgraph induced by E^2 is called a *pre-fragment*. We also introduce *elementary* pre-fragments $(\{v\}, \emptyset)$, where v is an ordinary 1-labelled node such that there is $e \in E^{\rm in}(v)$ that does not belong to $Z_s(v) \cup (B(v) - B_s(v))$ for any $s \in \langle T \rangle$ tight for v. One can prove the following structural properties, which are analogous to ones of the "Hungarian tree with blossoms" in matching problems:

(4.11) each $e = uv \in Z$ with $u \in T$ is labelled from u to v and not from v to u;

(4.12) if v is 1-labelled then each (v, e, e') with $e \in E^{\rm in}(v)$ and $e' \in E^{\rm out}(v)$ and none with $e, e' \in E^{\rm in}(v) \cup E^{\rm un}(v)$ forms a fork;

(4.13) for each pre-fragment F all feasible edges in $\delta(VF)$ are labelled as leaving F except one, denoted by e_F, labelled as entering F.

(Proofs of (4.12) and (4.13) are based on the following easy corollary from (4.10):

(4.14) for $v \in VG^p$ and $e, e', e'' \in E(v)$, if neither (v, e, e') nor (v, e', e'') is a fork then (v, e, e'') is not a fork either.)

Our goal is to find β' and γ' as in (ii) in Theorem 4.1. Each pre-fragment F generates the fragment ϕ with X_ϕ to be the preimage of VF in G^p and

(4.15) $U_\phi = (B \cap \delta(X_\phi)) \cup \{e_F\}$ if $e_F \in Z$, and $U_\phi = (B \cap \delta(X_\phi)) - \{e_F\}$ if $e_F \in B$

(for e_F as in (4.13)). Since B is inner Eulerian and $VF \cap T = \emptyset$, ϕ is well-defined; also ϕ is saturated by B, and $e_F = r_\phi$. The set of such fragments is denoted by $\mathcal{F}^{\rm new}$; these are to be added to $\widehat{\mathcal{F}}$ when β will change. Properties (A0),(A1) and (A3) are easy for $\widehat{\mathcal{F}} \cup \mathcal{F}^{\rm new}$, while (A2) follows from the observation that if a non-elementary pre-fragment F would contain a fragment-node $v_{\phi'}$ incident to e_F with $e_F \neq r_{\phi'}$, then (by (4.10)(ii)) no edge in $E(v_{\phi'})$ could be labelled in both directions, contrary to the definition of F.

Let $\mathcal{F}^+$ ($\mathcal{F}^-$) be the set of $\phi \in \mathcal{F}^{\max}$ such that v_ϕ is 1-labelled and $r_\phi \in E^{\rm in}(v_\phi)$ (respectively, $r_\phi \in E^{\rm out}(v_\phi)$). Let $\mathcal{F}' = \mathcal{F}^+ \cup \mathcal{F}^{\rm new}$. We are going to transform β into $\beta' = \beta^\varepsilon$ by increasing β by ε on $\mathcal{F}'$ and decreasing it by ε on $\mathcal{F}^-$, with some $\varepsilon \in \mathbb{Q}_+$ satisfying $0 < \varepsilon \le \min\{\beta_\phi : \phi \in \mathcal{F}^-\}$ (which ensures the nonnegativity of β').

We explain that one can choose a (sufficiently small) $\varepsilon > 0$ and transform γ into $\gamma' = \gamma^\varepsilon$ so that the new length function $\ell' = \ell^\varepsilon := a + \gamma' + \sum(\beta'_\phi \chi_\phi : \phi \in \mathcal{F})$, number $p' = p^\varepsilon := \min\{{\rm dist}_{\ell'}(s,t) : s, t \in T, s \neq t\}$ and the graph $G^{p'}$ (concerning ℓ') satisfy:

(4.16) $$p' = p + 2\varepsilon;$$

(4.17) $$G^{p'} \text{ contains } B.$$

First, we partition VG^p into four sets T, L, M, W, where L consists of the ordinary 1-labelled nodes not forming pre-fragments, M consists of the preimages of unlabelled elements of $VG^* - T$, and $W = VG^p - (T \cup L \cup M)$ (then the sets X_ϕ for $\phi \in \mathcal{F}' \cup \mathcal{F}^-$ give a partition of W).

Second, fix some edge h_v in $E^{\rm in}(v)$ for each 1-labelled node v. For $v \in VG^p$ and $e \in E(v)$, define the number $\rho(v, e) = \rho^\varepsilon(v, e)$ as follows:

(4.18)(i) if $v \in W \cup M$, set $\rho(v,e)=0$;

(ii) if $v \in T$, set $\rho(v,e)=\varepsilon$;

(iii) if $v \in L$ and $e \in Z$, set $\rho(v,e)=\varepsilon$ if (v,h_v,e) is a fork, and $-\varepsilon$ otherwise;

(iv) if $v \in L$ and $e \in B$, set $\rho(v,e)=-\varepsilon$ if (v,h_v,e) is a fork, and ε otherwise.

(Using (4.12) and (4.14), one shows that $\rho(v,e)$ does not depend on the choice of h_v; recall that (v,h_v,h_v) is not a fork.) Next, for $e=uv \in EG^p$ define

$$\rho(e)=\rho(u,e)+\rho(v,e), \tag{4.19}$$

and then define γ' by

$$\begin{aligned} \gamma'_e &= \gamma_e+\rho(e)+\widehat{\beta}(e)-\widehat{\beta}'(e) \qquad \text{for } e \in B, \\ &= 0 \qquad \text{for the remaining edges } e \text{ in } G; \end{aligned} \tag{4.20}$$

where $\widehat{\beta}(e)=\sum(\beta_\phi\chi_\phi(e) \,:\, \phi \in \mathcal{F})$ and $\widehat{\beta}'(e)=\sum(\beta'_\phi\chi_\phi(e) \,:\, \phi \in \mathcal{F})$. Obviously, (4.1) holds for γ'.

The proof that the B,β',γ' have the desired properties falls into several stages. First of all one shows that (β',γ') is p'-admissible and B is regular for p',ℓ'. This follows from three lemmas.

Lemma 4.4. *For some (and each smaller) $\varepsilon>0$, γ^ε and ℓ^ε are nonnegative.*

Lemma 4.5. *Let $P=(v_0,e_1,v_1,\dots,e_k,v_k)$ be a path in a packing $\mathcal{D}$ representing B. Then $\ell^\varepsilon(P)=p+2\varepsilon$ for some (and each smaller) $\varepsilon>0$.*

Lemma 4.6. *(4.16) holds for some (and each smaller) $\varepsilon>0$.*

To prove Lemma 4.4, it suffices to consider an edge $e=uv \in B$ with $\gamma_e=0$. The proof depends on occurrencies of u and v in T,L,W,M. Considering possible cases, one can obtain the important property that

$$\text{if } e \text{ is labelled then } \rho(e)+\widehat{\beta}(e)-\widehat{\beta}'(e)=0, \tag{4.21}$$

whence $\gamma'_e=\gamma_e=0$ (e.g., if e is labelled from u to v and $u,v \in T \cup L$, then $\widehat{\beta}'(e)=\widehat{\beta}(e)=0$, and (4.18) shows that $\rho(u,e)=-\varepsilon$ and $\rho(v,e)=\varepsilon$).

If e is unlabelled then, in view of (4.12) and (4.13), the possible cases are: (i) $u,v \in T \cup L$; (ii) $u \in T \cup L$ and $v \in W \cup M$; (iii) $u,v \in M$ or $u,v \in X_\phi$ for some $\phi \in \widetilde{\mathcal{F}} \cup \mathcal{F}^{\mathrm{new}}$; (iv) $u \in X_\phi$, $e \in U_\phi$ for some $\phi \in \mathcal{F}^-$ and either $v \in M$ or $v \in X_{\phi'}$ for some $\phi' \in \mathcal{F}^- - \{\phi\}$. In case (i), $\rho(u,e)=\rho(v,e)=\varepsilon$; in case (ii), $\rho(u,e)=\varepsilon$, $\rho(v,e)=0$ and $\widehat{\beta}(e)-\widehat{\beta}'(e) \ge -\varepsilon$; in case (iii), $\rho(e)=\widehat{\beta}(e)-\widehat{\beta}'(e)=0$; in case (iv), $\rho(e)=0$ and $\widehat{\beta}(e)-\widehat{\beta}'(e) \in \{\varepsilon,2\varepsilon\}$. Thus, $\gamma'_e \ge \gamma_e$ holds in all cases.

To prove Lemma 4.5, put $q_i = \rho(v_i, e_i) + \rho(v_i, e_{i+1})$, $i = 1, \dots, k-1$. Then, by (4.20) and (4.18)(ii),

$$\ell^\varepsilon(P) - \ell(P) = \sum_{i=1}^{k} \rho(e) = \rho(v_0, e_1) + \sum_{i=1}^{k-1} q_i + \rho(v_k, e_k) = 2\varepsilon + \sum_{i=1}^{k-1} q_i.$$

We observe that $q_i = 0$ for each i. Indeed, if $v_i \in W \cup M$ then $\rho(v_i, e_i) = \rho(v_i, e_{i+1}) = 0$. And if $v = v_i \in L$ then the the facts that $P \in \mathcal{D}$ and that v does not form an elementary pre-fragment imply that (i) $h_v \in Z_s \cup (B(v) - B_s(v))$ for some $s \in \langle T \rangle$ tight for v, and (ii) one of e_i, e_{i+1} is in $B_s(v)$ and the other in $B(v) - B_s(v)$. Therefore, $\rho(v, e_i) = -\rho(v, e_{i+1})$, whence $q_i = 0$.

In the proof of Lemma 4.6, we may assume that G^p contains at least one T-path. Then $B \neq \emptyset$ (otherwise there would exist an augmenting path), so there is a T-path P with $\ell^\varepsilon(P) = p + 2\varepsilon$ (by Lemma 4.5). We have to show that for some $\varepsilon > 0$, $\ell^\varepsilon(P) \geq p + 2\varepsilon$ holds for every T-path $P = (v_0, e_1, v_1, \dots, e_k, v_k)$ in G. This is obvious if $\ell(P) > p$. And if $\ell(P) = p$ then P lies in G^p. Then, arguing as in the proof of Lemma 4.5, one shows that $q_i \geq 0$ for $i = 1, \dots, k-1$. Furthermore, one shows that $\ell_e^\varepsilon \geq \ell_e + \rho^\varepsilon(e)$ for any $e \in EG^p$. This implies $\ell^\varepsilon(P) \geq p + 2\varepsilon$.

To complete the proof for case (ii) of Theorem 4.1, we have to show maintaining (A0)-(A6) for (β', γ'). Some of these properties (e.g., (A1)-(A3)) are easy, while the other ones take more efforts to prove. One of the most technical part of the entire proof is to show maintaining (A6). Next, it turns out that, under the above transformation of (β, γ), condition (A0) may not remain true for some $\phi \in \widehat{\mathcal{F}}$ with $X_\phi \subseteq M$. If this happens, one applies an additional transformation of β' on some ϕ's with $X_\phi \subseteq M$ and of γ' on some edges incident to nodes in M, after which (A0) becomes true. We do not go into details of such a transformation, referring the reader to [22].

The proof of Theorem 4.1 provides an algorithm to solve (1.7). At each iteration, either (i) the value $\mathrm{val}(B)$ of the current regular set B increases, or (ii) the current (β, γ) is transformed so as to increase p. The number of iterations of type (i) is $\nu \leq |EG|$. Each iteration of type (ii) finds ε as large as possible provided that the resulting β', γ', ℓ' are nonnegative and $p' = p^\varepsilon$ equals $p + 2\varepsilon$. If such an ε is unbounded, the current B gives an o.s. to (1.7) for every sufficiently large p (so it solves (1.3) with $H = K_T$ and $c = 1\!\!1$). One shows that the maximal choice of ε ensures that the number of consecutive iterations of type (ii) is $O(|VG|^2)$. As a consequence, the total number of iterations is polynomial in $|VG|$ and the running time of the algorithm is strongly polynomial.

5. (T', d)-joins

We outline the proof of Theorem 5 given in [3], using corresponding notations from Sections 1,2 and 4.

To see $\mathrm{conv}(\mathcal{B}_d) \subseteq D'$, observe that the incidence vector $x' = \xi_{B'}$ of each (T', d)-join B' is contained in D'. Indeed, (i) and (iii) in (1.16) are obvious, and (ii) easily follows by considering corresponding T'-paths in (VG', B') that realize the demand d. A reverse property is also true; namely, for each integer vector x' in D', $B' = \{e : x'_e = 1\}$ is a (T', d)-join. Indeed, if $\deg_{B'}(v) \not\equiv d_v \pmod 2$ for some $v \in VG'$, then the pair $\phi = (\{v\}, B'(v))$ forms a fragment for G', T', d (cf. (1.15)). But $x'\chi_\phi = |B'(v)| = |U_\phi|$, contradicting (1.16)(iii). Consider the d-extension G of G' (defined in the Introduction), and let $T = \{s : s' \in T'\}$ and $B = B' \cup (EG - EG')$. Then B is inner Eulerian for G, T. Furthermore, (1.16)(ii) implies that for each $s \in T$ and $Y \subset VG$ with $Y \cap T = \{s\}$, the cut $\delta^G(Y)$ meets at least $d_{s'}$ edges of B. So, by Lovász-Cherkassky's theorem mentioned in Example 3 in the Introduction, the graph (VG, B) contains $d(T')/2$ edge-disjoint T-paths. Hence, B' is a (T', d)-join.

To prove $D' \subseteq \mathrm{dom}(\mathcal{B}_d)$, it suffices to show that (i) if $\mathcal{B}_d = \emptyset$ then $D' = \emptyset$, and (ii) if $\mathcal{B}_d \neq \emptyset$ then the problem

(5.1) *given a weighting* $w : EG' \to \mathbb{Z}_+$, *minimize* wx' *over all* $x' \in D'$,

has an *integer* o.s. x'. Indeed, in case (i), we have $\mathrm{dom}(\mathcal{B}_d) = \emptyset + \mathbb{R}_+^{EG'} = \emptyset = D'$, while in case (ii), varying w, we conclude that all vertices of $D' + \mathbb{R}_+^{EG'}$ are the incidence vectors of (T', d)-joins, whence $D' \subseteq \mathrm{dom}(\mathcal{B}_d)$.

Suppose that $\mathcal{B}_d$ is nonempty. Consider $\mathcal{D}, \beta, \gamma$ that achieve the equality in (1.12) for G, T, a and a rather large positive p, where G is the d-extension of G', $a_e = w_e$ for $e \in EG'$ and $a_e = 0$ for the edges of G connecting T and T'. Let $x \in \mathbb{R}^{EG}$ be the incidence vector of the set of edges occurred in $\mathcal{D}$. Then the restriction x' of x to EG' is the incidence vector of a (T', d)-join; therefore, $x' \in D'$. Take an arbitrary $y' \in D'$ and extend y' by ones to the edges connecting T and T', denoting the resulting vector in $\mathbb{R}^{EG}$ by y. We assert that $ay \geq ax$ (whence $wy' \geq wx$, and the result follows).

Let $\overline{d} = d(T)/2$, $\overline{\beta} = \sum(\beta_\phi \chi_\phi : \phi \in \mathcal{F}^0)$ and $\ell = a + \gamma + \overline{\beta}$ (cf. (1.9)). By (1.12), we have

$$p\overline{d} - ax = \psi(p, \mathcal{D}) = \gamma(EG) + \sum(\beta_\phi(|U_\phi| - 1) : \phi \in \mathcal{F}^0). \tag{5.2}$$

On the other hand, from (1.16)(ii) for y' and Lovász-Cherkassky's theorem one can deduce that there exist T-paths $P^1, \ldots, P^k$ in G and reals $\lambda_1, \ldots \lambda_k \geq 0$ such that $\lambda_1 + \ldots + \lambda_k = \overline{d}$ and $\sum(\lambda_i \xi_{P^i} : i = 1, \ldots, k) \leq y$, where ξ_{P^i} is the incidence vector of the edge set of P^i. By (1.11), $\ell(P^i) \geq p$. Therefore, $\ell y \geq p\overline{d}$ (as ℓ is nonnegative), and we obtain

$$p\overline{d} - ay \leq \gamma y + \overline{\beta} y. \tag{5.3}$$

Furthermore, from (1.16)(iii) for y' one can deduce that $\chi_\phi y \leq |U_\phi| - 1$ for each inner fragment ϕ for G, T. Hence, $\overline{\beta} y \leq \sum(\beta_\phi(|U_\phi| - 1) : \phi \in \mathcal{F}^0)$. Now comparing (5.2) and (5.3) gives the desired inequality $ay \geq ax$.

Using similar arguments for the number $p = |EG'| + 1$ and weights $w_e = 1$ for all $e \in EG'$, one shows that $D' \neq \emptyset$ implies $\mathcal{B}_d \neq \emptyset$, giving (i) as above.

A similar method is applied to derive Theorem 6 from Theorem 3 (see [3]).

6. Generalizations, open problems

Generalizing (1.4), Mader [32] established a minimax relation that expresses the maximum number of pairwise openly (node) disjoint T-paths (see also [29,24]). It turns out that Theorem 3 can also be extended to the openly disjoint case. We deal with an edge cost function a as before, and consider the problem:

(6.1) *given $p \in \mathbb{Q}_+$, maximize the objective $\psi(p, \mathcal{D}) = p|\mathcal{D}| - a(\mathcal{D})$ among all sets of pairwise openly disjoint T-paths in G.*

To state a minimax relation involving ψ, we need some definitions and notations.

A. For $E \subseteq EG$, let $\nabla(E)$ denote the set of nodes incident to edges in both E and $EG - E$.

B. A triple $\phi = (X_\phi, E_\phi, A_\phi)$ is called a *graph-fragment* if (X_ϕ, E_ϕ) is a connected subgraph of G with $X_\phi \subseteq VG - T$, $A_\phi \subseteq \nabla(E_\phi)$, and $|A_\phi|$ is an odd ≥ 3. The set of graph-fragments is denoted by $\mathcal{V}$. For $\phi \in \mathcal{V}$ let $\overline{A}_\phi$ stand for $\nabla(X_\phi) - A_\phi$. The *characteristic function* of ϕ is defined on the edges $e = uv \in EG$ by

$$
\begin{aligned}
\chi_\phi(e) &= 1 \quad \text{if } u \in A_\phi \text{ and } v \notin X_\phi, \\
&= -1 \quad \text{if } u \in \overline{A}_\phi \text{ and } v \notin X_\phi, \\
&= 2 \quad \text{if } u, v \in A_\phi \text{ and } e \notin E_\phi, \\
&= -2 \quad \text{if } u, v \in \overline{A}_\phi \text{ and } e \notin E_\phi, \\
&= 0 \quad \text{otherwise.}
\end{aligned}
$$

C. We say that a T-path $P = (v_0, e_1, v_1, \ldots, e_k, v_k)$ *touches* a graph-fragment ϕ at a node $v \in X_\phi$ if for some $0 < i < k$, $v = v_i$ and both e_i, e_{i+1} are not in E_ϕ. The number of indices i such that $v_i \in A_\phi$ and P touches ϕ at v_i is denoted by $\omega(P, \phi)$.

D. For a function γ on VG and an edge $e = uv$, let $\overline{\gamma}_e$ denote $\gamma_u + \gamma_v$. For a function β on $\mathcal{V}$, let $\overline{\beta}$ denote the function $\sum(\beta_\phi \chi_\phi : \phi \in \mathcal{V})$ on EG.

E. Given $\beta : \mathcal{V} \to \mathbb{Q}_+$ and $\gamma : VG \to \mathbb{Q}_+$, define function $\ell = \ell^{\beta,\gamma}$ on EG to be $a + \overline{\beta} + \overline{\gamma}$ and call the pair (β, γ) *p-admissible* if:

(i) $\gamma_v = 0$ for all $v \in T$;

(ii) ℓ is nonnegative;

(iii) the ℓ-length of every T-path P is at least $p + 2\sum(\beta_\phi \omega(P, \phi) : \phi \in \mathcal{V})$.

Without loss of generality we assume that no edge e of G connects two terminals (as we can replace such an e by two edges e', e'' in series with $a_{e'} + a_{e''} = a_e$).

Theorem 6.1 [23]. $\max\{\psi(p,\mathcal{D})\} = \min\{2\gamma(VG) + \sum(\beta_\phi(|A_\phi|-1) : \phi \in \mathcal{V})\}$, *where $\mathcal{D}$ runs over all sets of pairwise openly disjoint T-paths and (β,γ) runs over all p-admissible pairs.*

In conclusion of this paper we raise several open questions.

1) Is it possible to construct a "purely combinatorial" strongly polynomial algorithm for finding a half-integrer o.s. to (1.2) with $H = K_T$?

2) Is it true that, for any integer $p \geq 0$, the minimum in (1.12) in Theorem 3 is achieved by β, γ that are half-integral?

3) Does there exist a "good characterization" of the dominant $\mathrm{dom}(\mathcal{B}_d)$ for the set of (T', d)-joins via an explicit system of linear inequalities rather than the Minkowsky sum as in Theorem 5? Are the left hand side coefficients in the minimal integer description of facets of $\mathrm{dom}(\mathcal{B}_d)$ bounded? Similar questions are raised for $\mathrm{dom}(\mathcal{B}^{\max})$. [To comparison: the perfect matching polytope of a graph has a "good description" via inequalities with all left hand side coefficients in $\{0,1\}$, due to classic results of Edmonds [8], but it was shown in [5] that such coefficients for some facets of the corresponding dominant can be large.]

4) Theorem 5 shows the integrality of every vertex of D' that remains a vertex in $D' + \mathbb{R}_+^{EG'}$; these vertices are exactly the incidence vectors of minimal (T', d)-joins. Moreover, the sets of integral point in D' and $\mathrm{conv}(\mathcal{B}_d)$ are the same, and we know that $D' = \mathrm{conv}(\mathcal{B}_d)$ if $d = 1$. However, as shown in [3], D' can strictly include $\mathrm{conv}(\mathcal{B}_d)$ in general. Two questions seem to be closely related: (i) is there a "good description" of $\mathrm{conv}(\mathcal{B}_d)$ via linear constraints, and (ii) what is the complexity status of (1.14) for an arbitrary weighting $w : EG' \to \mathbb{Z}$? Similar questions arise for maximum multi-joins.

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