HALF-INTEGRAL FLOWS IN A PLANAR GRAPH WITH FOUR HOLES

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Abstract. Suppose that $G = (VG, EG)$ is a planar graph embedded in the euclidean plane, that I, J, K, O are four of its faces (called *holes* in G), that $s_1, ..., s_r, t_1, ..., t_r$ are vertices of G such that each pair $\{s_i, t_i\}$ belongs to the boundary of some of I, J, K, O, and that the graph $(VG, EG \cup \{\{s_1, t_1\}, ..., \{s_r, t_r\}\})$ is eulerian.

We prove that if the multi(commodity)flow problem in G with unit demands on the values of flows from s_i to t_i, $i = 1, \ldots, r$, has a solution then it has a *half-integral* solution as well. In other words, there exist paths $P_1^1, P_1^2, P_2^1, P_2^2, ..., P_r^1, P_r^2$ in G such that each P_i^j connects s_i and t_i, and each edge of G is covered at most twice by these paths. (It is known that in case of at most three holes there exist edge-disjoint paths connecting s_i and t_i, $i = 1, \ldots, r$, provided that the corresponding multiflow problem has a solution, but this is, in general, false in case of four holes.)

Keywords. Planar graph, Edge-disjoint paths, Cut, Metric.

1. Introduction

Throughout, we deal with an undirected planar graph G; speaking of a planar graph we mean that some its embedding in the euclidean plane $\mathbb{R}^2$ (or the sphere) is fixed. VG is the vertex set, EG is the edge set of G (multiple edges and loops are admitted), and $\mathcal{F} = \mathcal{F}_G$ is the set of faces of G. A subset $\mathcal{H} \subseteq \mathcal{F}$ of faces of G, called its *holes*, is distinguished. Let $U = \{\{s_1, t_1\}, ..., \{s_r, t_r\}\}$ be a family of pairs (possibly repeated) of vertices of G such that each $\{s_i, t_i\}$ is contained in the boundary $bd(I)$ of some hole $I \in \mathcal{H}$.

Problem (G, U, k): *given an integer* $k \geq 1$*, find* $P_1^1, \ldots, P_1^k, P_2^1, \ldots, P_2^k, \ldots, P_r^1, \ldots, P_r^k$ *such that each* P_i^j *is a path in* G *connecting* s_i *and* t_i*, and each edge of* G *occurs at most* k *times in these paths.*

If no restriction on k is imposed, the problem is denoted as $(G, U)^*$; thus $(G, U)^*$ is the fractional relaxation of $(G, U, 1)$, or the *multi*(commodity)*flow* problem with unit capacities of the edges of G and unit demands on flows connecting the pairs in U.

We prove the following theorem.

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Theorem 1. *Let $|\mathcal{H}| = 4$, and let the graph $(VG, EG \cup U)$ be eulerian, that is,*

$$(1.1) \qquad |\delta X| + |\{i : \delta X \text{ separates } s_i \text{ and } t_i\}| \text{ is even for any } X \subset VG.$$

Let $(G, U)^*$ have a solution. Then $(G, U, 2)$ has a solution as well; in other words, there exist $P_1^1, P_1^2, P_2^1, P_2^2, \dots, P_r^1, P_r^2$ such that each P_i^j is a path in G connecting s_i and t_i, and each edge of G is covered at most twice by these paths.

[For $X \subseteq V$, $\delta X = \delta^G X$ denotes the set of edges of G with one end in X and the other in $VG - X$; a nonempty set δX is called a *cut* in G; we say that δX *separates* vertices x and y if exactly one of x, y is in X.] An obvious necessary condition for solvability of (G, U, k) for arbitrary G, U, k is the *cut condition*:

(1.2) each cut δX in G separates at most $|\delta X|$ pairs in U.

The following result is well known.

Okamura's theorem [Ok]. *If $|\mathcal{H}| = 2$ and if (1) and (2) hold then the problem $(G, U, 1)$ has a solution (that is, there exist edge-disjoint paths $P_1, \dots, P_r$ in G such that P_i connects s_i and t_i).*

(An analogous result for $|\mathcal{H}| = 1$ was stated in [OkS].) The cut condition is, in general, not sufficient for the solvability of (G, U, k) when $|\mathcal{H}| = 3$. However, the following is true.

Theorem 2 [Ka2]. *Let $|\mathcal{H}| = 3$, and let (1) hold. The problem $(G, U, 1)$ has a solution if (2) and the following 2,3-metric condition hold:*

$$(1.3) \qquad \sum_{e \in EG} m(e) \geq \sum_{i=1}^{r} m(s_i, t_i) \qquad \text{for all 2,3-metrics } m \text{ on } VG.$$

[By a *metric* on a set V we mean a real-valued function m on $V \times V$ satisfying $m(x, x) = 0$, $m(x, y) = m(y, x)$ and $m(x, y) + m(y, z) \geq m(x, z)$ for all $x, y, z \in V$. We say that m is *induced* by (H, σ), where H is a graph and σ is a mapping of V into VH, if $m(x, y)$ is equal to $\mathrm{dist}^H(\sigma(x), \sigma(y))$ for all $x, y \in V$; here $\mathrm{dist}^{G'}(x', y')$ denotes the distance in a graph G' between vertices x' and y'. When it leads to no confusion, we may say that m is induced by H or m is induced by σ. If $\sigma(V) = VH$ and H is the complete graph K_2 on two vertices (the complete bipartite graph $K_{2,3}$ with parts of two and three vertices) then m is said to be a *cut metric* (respectively, a 2,3-*metric*).] To satisfy the inequality in (3) for any metric m on VG is necessary for the solvability of (G, U, k) for arbitrary G, U, k because if P_i^j's as above give a solution of (G, U, k) then

$$\sum_{e \in EG} m(e) \geq \frac{1}{k} \sum_{i=1}^{r} \sum_{j=1}^{k} \sum (m(e) : e \in P_i^j) \geq \sum_{i=1}^{r} m(s_i, t_i),$$

since m satisfies the triangle inequalities (here we write $e \in P_i^j$ considering a path as an edge-set). Thus, if $|\mathcal{H}| \le 3$, (1) holds, and $(G,U)^*$ has a solution then $(G,U,1)$ has a solution as well. It turned out that such a property does not remain, in general, true for $|\mathcal{H}| = 4$, as shown in [Ka2]. Hence, when $|\mathcal{H}| = 4$, Theorem 1 gives the least (in terms of $\mathcal{H}$) value of k for which (G,U,k) has a solution in the eulerian case.

Another difference between cases $|\mathcal{H}| = 3$ and $|\mathcal{H}| = 4$ is that in the latter case more exotic metrics are involved in solvability conditions for $(G,U)^*$.

Theorem 3 [Ka1]. *For $|\mathcal{H}| = 4$, $(G,U)^*$ is solvable if and only if (1.3) holds for every m that is a cut metric or a 2,3-metric or a metric induced by a bipartite planar graph H with $|\mathcal{F}_H| = 4$.*

To prove Theorem 1, we need to strengthen this result and establish, in Theorems 4 and 3.11 below, a number of additional properties of the latter kind of metrics mentioned in Theorem 3. More precisely, suppose we are given a function $c : EG \to \mathbb{Q}_+$ (of *capacities* of edges) and numbers $d_1, \dots, d_r \in \mathbb{Q}_+$ (*demands*). Denote by $\mathcal{P}_i = \mathcal{P}(G, s_i, t_i)$ the set of (simple) paths from s_i to t_i (or $s_i - t_i$ *paths*) in G. Let $\mathcal{P}(G,U) := \cup(\mathcal{P}_i : i = 1, \dots, r)$. The *multiflow (multicommodity flow) problem* for c and d consists in finding a function $f : \mathcal{P}(G,U) \to \mathbb{Q}_+$ satisfying:

$$(1.4) \qquad f^e := \sum (f(P) : e \in P \in \mathcal{P}(G,U)) \le c(e) \quad \text{for all } e \in EG;$$

$$(1.5) \qquad \sum (f(P) : P \in \mathcal{P}_i) = d_i \quad \text{for } i = 1, \dots, r.$$

This problem is denoted by (c,d), and an f satisfying (1.4)- (1.5) is called a (c,d)-admissible *multiflow*. Applying Farkas lemma, we obtain the following

Criterion of solvability of (c,d) (for arbitrary G and U): (c,d) is solvable if and only if

$$(1.6) \qquad \sum_{e \in EG} c(e) l(e) \ge \sum_{i=1}^{r} d_i \mathrm{dist}_l(s_i, t_i)$$

holds for any function $l : EG \to \mathbb{Q}_+$

(cf. [Iri,KeO]); here $\mathrm{dist}_l(x,y)$ denotes the distance between vertices x and y in G whose edges e have *lengths* $l(e)$. Let us say that l is *bipartite* if l is integer-valued and the length $l(C) := \sum(l(e) : e \in C)$ of every circuit C in G is even. Clearly in the above criterion it suffices to require satisfying (1.6) only for the bipartite l's. In what follows we consider only bipartite l's.

For G and H as above let $W(\mathcal{H})$ denote the set of pairs $\{s,t\}$ of vertices such that s,t are in the boundary $\mathrm{bd}(I)$ of some $I \in \mathcal{H}$.

Definition. Let $l, l' : EG \to \mathbb{Z}_+$ be bipartite, $l' \le l$ and $l' \ne 0, l$. We say that l' *$\mathcal{H}$-reduces* l if

$$(1.7) \qquad \mathrm{dist}_l(s,t) = \mathrm{dist}_{l'}(s,t) + \mathrm{dist}_{l-l'}(s,t) \quad \text{for each } \{s,t\} \in W(\mathcal{H}).$$

A triple $(G, \mathcal{H}, l)$ is called *primitive* if there is no l' that $\mathcal{H}$-reduces l.

It is easy to see that in the above criterion it suffices to consider only those l''s for which $(G, \mathcal{H}, l)$ is primitive. We prove the following theorem.

Theorem 4. *Let $|\mathcal{H}| = 4$, and let $(G, \mathcal{H}, l)$ be primitive. Then $l(e) \le 4$ for all $e \in EG$.*

Note that if $|\mathcal{H}| \le 2$ then, in view of Okamura's theorem, for every primitive $(G, \mathcal{H}, l)$, l corresponds, in a sense, to a certain cut, and hence, $l(e) \le 1$ for all $e \in EG$, while if $|\mathcal{H}| = 3$ then, in view of Theorem 2, for every primitive $(G, \mathcal{H}, l)$, l corresponds to a cut or a 2,3-metric, so $l(e) \le 2$ for all $e \in EG$.

The paper has the following structure. In Section 2 we prove the existence of a 1/4-integral solution (in the eulerian case), which is relatively easy and is based on Okamura's theorem and a strengthening of the corresponding fractional version of Theorem 2 (Theorem 2' below). Section 3 is devoted to a proper study of primitive $(G, \mathcal{H}, l)$ for $|\mathcal{H}| = 4$. Here we prove Theorem 4 and establish certain additional properties for the case when there is $e \in EG$ with $l(e) = 4$ (Theorem 3.11); this results will be one of the main tools in the further proof of Theorem 1. The proof of Theorem 1 will consists of three stages described in Sections 4-6.

Throughout the paper, the faces of a planar graph G' in question are considered as *open* regions in the plane. An edge of G' is identified with the corresponding curve without the end points in $\mathbb{R}^2$. When it leads to no confusion, an edge $e \in EG'$ with end vertices x and y may be denoted by xy; a path (circuit) $P = (x_0, e_1, x_1, \dots, e_k, x_k)$ (where $x_i \in VG'$ and $e_i \in EG'$) may be denoted by $x_0x_1 \dots x_k$; a path P is identified with its image in the plane (note that P will be often considered up to reversing and/or cyclical shifting). $|P|$ denotes the *length* (i.e. the number of edges) of a path P. If $|P| = 0$, P is called *trivial*. The boundary $\mathrm{bd}(F)$ of a face $F \in \mathcal{F}_{G'}$ is identified with the corresponding (possibly not simple) circuit in G'.

As mentioned above, there is a strengthening of the fractional version of Theorem 2; in particular, it demonstrates a topologic correspondence, in a sense, of "primitive 2,3-metrics" for $G', \mathcal{H}'$ with $|\mathcal{H}'| = 3$ to the face structure of $K_{2,3}$.

Fig. 1.1

Theorem 2' [Ka1]. *Let $|\mathcal{H}'| = 3$, $c' : EG' \to \mathbb{Q}_+$ and $d' : U' \to \mathbb{Q}_+$, and let the problem (c', d') have no solution. Then there exists m that is a cut- or 2,3-metric on VG' such that*

$$(1.8) \qquad c'(m) := \sum_{e \in EG'} c'(e)m'(e) < \sum_{\{s,t\} \in U'} d(s,t)m(s,t) =: d'(m).$$

Moreover, if m is a 2,3-metric then it is induced by $\sigma : VG' \to VK_{2,3}$ for which the following properties hold: if $\{x_1, x_2, x_3\}$ and $\{y_1, y_2\}$ are the parts in $K_{2,3}$ and $\Pi(\sigma)$ is the (ordered) partition $(S_1, S_2, S_3, T_1, T_2)$ of VG', where $S_i := \sigma^{-1}(x_i)$ and $T_j := \sigma^{-1}(y_j)$, then for some labelling $I_1, I_2, I_3 = I_0$ of the members of $\mathcal{H}'$:

(1.9) all sets in $\Pi(\sigma)$ are nonempty; for $i = 1, 2, 3$ the subgraph $\langle S_i \rangle$ in G' induced by S_i is connected; and $S_i \cap \mathrm{bd}(I_p) = \emptyset$ if and only if $p = i$;

(1.10) the space $\Omega(\sigma) := \mathcal{R}^2 - (I_1 \cup I_2 \cup I_3 \cup \Phi(S_1) \cup \Phi(S_2) \cup \Phi(S_3))$ consists of two disjoint regions, one containing T_1 and the other containing T_2; here $\Phi(S_i)$ is the union of $\langle S_i \rangle$ and the faces F of G' such that $\mathrm{bd}(F) \subseteq \langle S_i \rangle$.

In particular, G' has no edges connecting T_1 and T_2.

(See Fig. 1.1.)

2. EXISTENCE OF A QUARTER-INTEGRAL SOLUTION

Let us fix some solution $f : \mathcal{P}(G,U) \to \mathbb{Q}_+$ for $(G,U)^* = (c,d)$, where c and d are the all-unit functions on EG and U, respectively. It is convenient to think of f as consisting of four flows f_I, f_J, f_K, f_O, where $\mathcal{H} = \{I, J, K, O\}$ and f_F if the restriction of f to the paths in $\mathcal{P}(G,U)$ with both ends in $\mathrm{bd}(F)$. Denote by $\mathcal{L} = \mathcal{L}(f)$ the set of paths $P \in \mathcal{P}(G,U)$ with $f(P) > 0$ (the *support* of f). Similarly, $\mathcal{L}_F = \mathcal{L}_F(f)$ denotes the support of f_F; thus $\{\mathcal{L}_I, \mathcal{L}_J, \mathcal{L}_K, \mathcal{L}_O\}$ is a partition of $\mathcal{L}$.

A path $P \in \mathcal{L}_F$ ($F \in \mathcal{H}$) divides the space $\mathbb{R}^2 - F$ into a pair $\mathcal{R}(P)$ of closed regions whose intersection is P and union is $\mathbb{R}^2 - F$. We say that f is *non- crossing* if any two paths $P \in \mathcal{L}_F$ and $P' \in \mathcal{L}_{F'}$ for $F \neq F'$ do not cross, that is, P' is contained entirely in one set in $\mathcal{R}(P)$. Applying to f standard uncrossing techniques, it is easy to show that

(2.1) if $(G,U)^*$ has a $1/k$-integral solution then it has a $1/k$-integral non-crossing solution.

In what follows we assume that f is non-crossing. Consider two holes $F, F' \in \mathcal{H}$. The hole F' lies in some component Z of the space obtained by removing from the plane the set $\mathrm{bd}(F)$ and the paths in $\mathcal{L}_F$. The closed set $\mathbb{R}^2 - Z$ is denoted by $\Psi_{FF'}$.

From the fact that every path in $\mathcal{L}_F$ is simple it easily follows that the boundary of $\Psi_{FF'}$ is a simple circuit; denote it by $C_{FF'}$. The definition of Z shows that

(2.2) for each edge $e \in C_{FF'}$ at least one of the following true: (i) $f_F^e > 0$, or (ii) $e \in \mathrm{bd}(F)$.

(Here f_F^e is $\sum(f(P) : e \in P \in \mathcal{L}_F(f))$.) Next, as f is non-crossing,

(2.3) the circuits $C_{FF'}$ and $C_{F'F}$ are not crossing, and $\Psi_{FF'} \cap \Psi_{F'F} = C_{FF'} \cap C_{F'F}$.

We say that $C_{FF'}$ *separates* holes I, I' if they lie in different components of $\mathbb{R}^2 - C_{FF'}$. Obviously, if $C_{F,F'}$ does not separate holes F' and F'' then $\Psi_{FF'} = \Psi_{FF''}$, and therefore $C_{FF'} = C_{FF''}$. Circuits $C_{FF'}$ and $C_{F'F}$ are called *neighbouring* if there is no hole $F'' \neq F, F'$ for which $C_{F''F}$ separates F and F''. A maximal set $B \subseteq \mathcal{H}$ such that the circuits $C_{FF'}$ and $C_{F'F}$ are neighbouring for any distinct $F, F' \in B$ is called a *bunch*. Clearly for any $F, F', F'' \in B$ the circuit $C_{FF'}$ coincides with $C_{FF''}$. Regarding B, the circuit $C_{FF'}$ (the region $\Psi_{FF'}$) may be denoted as C_F (resp., Ψ_F); the family of the $|B|$ circuits C_F, $F \in B$, is denoted by $\mathcal{C}(B)$ (note that C_F and $C_{F'}$ may coincide for some $F \neq F'$). Summing up above observations we have:

(2.4) if B is a bunch then:

(i) the regions $\Psi_F, \Psi_{F'}$ are openly disjoint for distinct $F, F' \in B$;

(ii) the space $\mathbb{R}^2 - \cup(\Psi_F : F \in B)$ contains no hole;

(iii) each edge $e \in E$ belongs to at most two circuits in $\mathcal{C}(B)$.

For $F \in B$ denote by G_F (resp., $\mathcal{H}_F$; U_F) the subgraph of G contained in Ψ_F (resp., the set of holes $F' \in \mathcal{H}$ in Ψ_F; the set of pairs $\{s,t\} \in U$ belonging to $\mathrm{bd}(F')$, $F \in \mathcal{H}_F$). In particular, $F \in \mathcal{H}_F$, but possibly $\mathcal{H}_F$ contains more holes. From (2.4) it follows that the sets $\mathcal{H}_F$, $F \in B$, give a partition of $\mathcal{H}$.

Statement 2.1. *Suppose that there is $F \in B$ such that C_F has no edges in common with any $C_{F'}$, $F' \in \mathcal{H} - \{F\}$. Then $(G,U)^*$ has a half-integral solution.*

Proof. Consider the problems $(G_F, U_F)^*$ and $(G', U')^*$, where $G' = (VG, EG - EG_F)$ and $U' = U - U_F$. For $F' \in \mathcal{H}_F$ every path in $\mathcal{L}_{F'}$ lies in G_F, and for $F' \in \mathcal{H} - \mathcal{H}_F$ every path $P \in \mathcal{L}_{F'}$ lies in G' (since P uses no edges in C_F); hence both problems are solvable. As $|\mathcal{H}_F| \leq 3$ and $|\mathcal{H} - \mathcal{H}_F| \leq 3$, we deduce from Okamura's theorem and Theorem 2 that each problem has a *half-integral* solution (not necessarily integral since (1.1) may be violated, e.g., for G', U'). These give a half-integral solution for $(G,U)^*$. •

In what follows we assume that a non-crossing f and a bunch $B = \{F_1, \dots, F_k\}$ for it are chosen so that:

(2.5) (i) $|B|$ is as great as possible;

(ii) $\sum((|\mathcal{H}_F|)^2 \,:\, F \in B)$ is minimum subject to (i);

(iii) the number of faces in $\mathbb{R}^2 - \cup(\Psi_F \,:\, F \in B)$ is maximum subject to (i)-(ii);

(iv) the value $\sum(f^e \,:\, e \in C_F, F \in B)$ is minimum subject to (i)-(iii).

In particular, a bunch B (for some f) with $\{|\mathcal{H}_F| \,:\, F \in B\} = \{1,1,1,1\}$ is preferable than one with $\{1,1,2\}$, and $\{2,2\}$ is preferable than $\{1,3\}$. Let $\overline{f}^e_F$ stand for $\sum(f^e_{F'} \,:\, F' \in \mathcal{H}_F)$.

Lemma 2.2. *For each $F \in B$ there exists a function h_F on EG_F so that:*

(i) $h_F(e) \in \{0, \frac{1}{2}, 1\}$ for each $e \in C_F$ and $h_F(e) = 1$ for the other edges e in G_F;

(ii) if e is a common edge for C_F and $C_{F'}$ $(F, F' \in B)$ then $h_F(e) + h_{F'}(e) \le 1$;

(iii) each problem (h_F, d_F) is solvable; here d_F is the all-unit function on U_F.

This lemma shows the existence of a 1/4-integral solution for $(G,U)^*$. Indeed, for each $F \in B$ the function $2h_F$ is integral, hence the problem $(2h_F, 2d_F)$ has a half-integral solution. So (h_F, d_F) has a 1/4-integral solution. In view of (ii), these solutions give us an admissible solution for $(G,U)^*$.

Proof of Lemma 2.2. Choose functions h_F $(F \in B)$ in such a way that (ii)-(iii) hold and the value $\gamma(h) := \sum_{F \in B} |Q_F|$ is as small as possible; where $Q_F = Q_F(h)$ is the set of edges $e \in C_F$ for which $h_F(e)$ is different from $0, \frac{1}{2}, 1$. Such functions exist since we can take as h_F the function that is the restriction of $\overline{f}_F$ to C_F and all-unit on the other edges in G_F. One has to prove that $\gamma(h) = 0$. Suppose that $\gamma(h) > 0$.

For $F \in B$ let Q^+_F (Q^-_F) be the set of edges $e \in Q_F$ with $h_F(e) > 1/2$ (resp., $h_F(e) < 1/2$). For $\varepsilon \in \mathbb{R}_+$ and $F \in B$ define the function h^ε_F as

$$
\begin{aligned}
(2.6) \qquad h^\varepsilon_F(e) &:= h_F(e) - \varepsilon \quad \text{if } e \in Q^+_F;\\
&:= h_F(e) + \varepsilon \quad \text{if } e \in Q^-_F;\\
&:= h_F(e) \quad \text{for the remaining } e\text{'s in } G_F;
\end{aligned}
$$

Take ε to be maximum provided that: (a) $\varepsilon \le h_F(e) - 1/2$ for $F \in B$ and $e \in Q^+_F$; (b) $\varepsilon \le 1/2 - h_F(e)$ for $F \in B$ and $e \in Q^-_F$; and (c) for each $F \in B$ the problem (h^ε_F, d_F) is solvable. It is easy to see that $h^\varepsilon_F(e) + h^\varepsilon_{F'}(e) \le 1$ for each edge e common for C_F and $C_{F'}$, $F, F' \in B$. Furthermore, clearly $\gamma(h^\varepsilon) \le \gamma(h)$, so $\gamma(h^\varepsilon) = \gamma(h)$, by the choice of h. By the maximal choice of ε, there is $F \in B$ such that for any $\Delta > \varepsilon$ the problem $(h^{\varepsilon'}_F, d_F)$ has no solution for some $\varepsilon < \varepsilon' \le \Delta$. Two cases are possible.

Case 1. $|\mathcal{H}_F| \le 2$. Then, by Okamura's theorem, for every $\varepsilon' > \varepsilon$ there is $X' \subset VG_F$ such that $h^{\varepsilon'}_F(X') < d_F(X')$, where for $c' : EG_F \to \mathbb{R}$, $c'(X')$ stands for $\sum(c'(e) \,:\, e \in \delta X')$ and $d_F(X')$ stands for $|\{\{s,t\} \in U_F \,:\, \delta X' \text{ separates } s \text{ and } t\}|$

(letting $\delta X' := \delta^{G_F} X'$). Hence, there is $X \subset VG_F$ such that

$$h_F^\varepsilon(X) = d_F(X) \text{ and } h_F^{\varepsilon'}(X) < d_F(X) \text{ for any } \varepsilon' > \varepsilon.$$

Moreover, obviously, we may assume that δX is a *simple* cut, i.e. δX meets at most twice the boundary of every face in G_F. In particular, $|\delta X \cap C_F| \le 2$ since C_F is the boundary of some face in G_F. Then $|\delta X \cap C_F| = 2$; let $\delta X \cap C_F = \{e, e'\}$. If $\{e, e'\} \cap Q_F^+ = \emptyset$ or if $\{e, e'\} \cap Q_F^- \neq \emptyset$, then obviously $h_F^{\varepsilon'}(X) \ge h_F^\varepsilon(X)$. Hence, either (i) $e, e' \in Q_F^+$, or (ii) one of e, e', say e, is in Q_F^+ and the other, e', is in $C_F - Q_F$. Note that $d_F(X)$ is an integer and $h_F^\varepsilon(e'')$ is an integer for each $e'' \in \delta X - \{e, e'\}$. Therefore,

$$\phi := h_F^\varepsilon(e) + h_F^\varepsilon(e') = d_F(X) - \sum(h_F^\varepsilon(e'') \,:\, e'' \in \delta X - \{e, e'\})$$

is an integer. But, in case (i), $1/2 < h_F^\varepsilon(e) < 1$ and $1/2 < h_F^\varepsilon(e') < 1$, and, in case (ii), $1/2 < h_F^\varepsilon(e) < 1$ and $h_F^\varepsilon(e) \in \{0, \frac{1}{2}, 1\}$, whence ϕ cannot be integral; a contradiction.

Case 2. $|\mathcal{H}_F| = 3$. Then $|B| = 2$; let $F = I$, $B = \{I, K\}$ and $\mathcal{H}_I = \{I, J, O\}$. Apply Theorem 2'. Arguing as above, we conclude that there exists (i) $X \subseteq VG_I$ such that $h_I^\varepsilon(X) = d_I(X)$ and $h_I^{\varepsilon'}(X) < d_I(X)$ for any $\varepsilon' > \varepsilon$, or (ii) a 2,3-metric m on VG_I such that

$$h_I^\varepsilon(m) = d_I(m) \text{ and } h_I^{\varepsilon'}(m) < d_I(m) \text{ for any } \varepsilon' > \varepsilon,$$

where, as in (1.8), $h_I^\varepsilon(m)$ stands for $\sum(h_I^\varepsilon(e)m(e) \,:\, e \in EG_I)$ and $d_I(m)$ stands for $\sum(m(s,t) \,:\, \{s,t\} \in U_I)$. If (i) takes place, we come to a contradiction as in Case 1 above.

Thus (ii) takes place. One may assume that m is induced by a mapping $\sigma : VG_I \to VK_{2,3}$ for which $\Pi(\sigma) = (S_1, S_2, S_2, T_1, T_2)$ is a partition as in Theorem 2'. Since C_I is the boundary of some face $\widetilde{F}$ in G_I and each subgraph $\langle S_i \rangle$ is connected, C_I can pass across exactly one component of $\Omega(\sigma)$, say the component Ω_1 that contains T_1. Next, if $e \in C_I$ is an edge connecting vertices $u \in S_i$ and $v \in S_j$ then we may replace it by a pair of edges in series, say $e' = uz, e'' = zv$, placing the vertex z in the region Ω_1 (it is easy to see that the new graph ${G'}_I$ and the corresponding metric m' on $V{G'}_I$ maintain the above properties). Thus, one may assume that each edge in C_I connecting different sets in $\Pi(\sigma)$ connects T_1 and some S_i. Let $\xi = (e_1 = u_1v_1, \ldots, e_k = u_kv_k)$ be the sequence of such edges in C_I, and the vertices $u_1, v_1, \ldots, u_k, v_k$ occur in this order in C_I. Notice that there are no two consecutive edges e_j, e_{j+1} in ξ such that $v_j, u_{j+1} \in T_1$ and $u_j, v_{j+1} \in S_i$ for some $i \in \{1, 2, 3\}$. For otherwise, assuming for definiteness that $i = 1$ and letting Z to be the set of vertices of the component in $\langle T_1 \rangle$ that contains the part of C_I from v_j to u_{j+1}, the partition $(T_1 - Z, T_2, S_1 \cup Z, S_2, S_3)$ defines a 2,3-metric m' such that $h_I^\varepsilon(m') < h_I^\varepsilon(m)$ and $d_F(m') = d_F(m)$, which is impossible. This and the fact that each $\langle S_i \rangle$ is connected imply that $k \le 6$ and for each $i = 1, 2, 3$ there is at most one edge e_j such that $u_j \in S_i$ and $v_j \in T_1$. Consider three cases.

(i) $k = 2$. Then a contradiction is shown in a similar way as in Case 1.

(ii) $k = 6$. Let for definiteness $v_1, u_2 \in S_1$, $v_3, u_4 \in S_2$ and $v_5, u_6 \in S_3$; see Fig. 2.1a. Denote by Z_1 (resp., Z_2; Z_3) the set of vertices in T_1 that lie in the component of the space $\Omega_1 - \widetilde{F}$ that contains the part of C_I from v_4 to u_5 (resp., from v_6 to u_1; from v_2 to u_3). Then $\{Z_1, Z_2, Z_3\}$ is a partition of T_1. Shrink S_i to a single vertex s_i, Z_i to a vertex z_i, and T_2 to a vertex t_2, obtaining the graph Γ as shown in Fig. 2.1b.

Fig. 2.1

Consider the natural mapping $\tau : VG_I \to V\Gamma$; let m' be the metric on VG_I induced by τ. It is easy to see that $m'(e) = m(e)$ for each edge $e \in EG_I$ and $m'(p,q) = m(p,q)$ for each $\{p,q\} \in U_I$. Finally, one can check that $m' = \rho_{X(1)} + \rho_{X(2)} + \rho_{X(3)}$, where for $i = 1,2,3$, $X(i) := \tau^{-1}(\{s_i, z_{i-1}, z_{i+1}\})$ (letting $z_4 = z_1$ and $z_0 = z_3$), and $\rho = \rho_{X'}$ denotes the cut metric on VG_I defined as $\rho(x,y) := 1$ if $|X' \cap \{x,y\}| = 1$, and $\rho(x,y) := 0$ otherwise. Then $h_I^\varepsilon(X(i)) = d_I(X(i))$, $i = 1,2,3$. Moreover, for at least one i we have $h_I^{\varepsilon'}(X(i)) < d_I(X(i))$ (for $\varepsilon' > \varepsilon$); a contradiction.

(iii) $k = 4$. Let for definiteness $v_1, u_2 \in S_1$ and $v_3, u_4 \in S_2$; see Fig. 2.2a. Let Z_1 (Z_2) be the subset of T_1 in the component of $\Omega_1 - \widetilde{F}$ that contains the part of C_I from v_2 to u_3 (resp., from v_4 to u_1).

Consider the mapping $\tau : VG_I \to VH$ that brings the sets $S_1, S_2, S_3, T_2, Z_1, Z_2$ to the vertices $s_1, s_2, s_3, t_2, z_1, z_2$ (respectively) of the graph H drawn in Fig. 2.2b. Let m' be the metric on VG_I induced by τ. Then $m'(e) = m(e)$ for each $e \in EG_I$ and $m'(p,q) = m(p,q)$ for each $\{p,q\} \in U_I$. This implies

$$h_I^\varepsilon(m') = d_I(m'). \tag{2.7}$$

Let f' be a solution of $(h_I^{\varepsilon'}, d_I)$ (f' concerns G_I). An easy consequence of the equality (2.7) is that any path $P \in \mathcal{L}(f')$ must be shortest for m'. On the other hand, it is easy to see that the vertex z_1 does not belong to any shortest path connecting vertices in $\tau(\mathrm{bd}(J))$ or in $\tau(\mathrm{bd}(O))$, while s_3 does not belong to any shortest path connecting vertices in $\tau(bd(I))$. This implies that the circuits $C_{JI}(f')$ and $C_{OI}(f')$ cannot separate I and K, while $C_{IJ}(f')$ cannot separate J and O. Now form a solution $\widehat{f}$ for $(G,U)^*$ by combining the flows f' and f_K. From said above it follows easily that for $\widehat{f}$ there is either a bunch B' such that either $|B'| \geq 3$, or $|B'| = 2$ and $\{|\mathcal{H}_F| : F \in B\} = \{2,2\}$. In each case B' contradicts with the choice of B in (2.5).

This completes the proof of the lemma. •

Fig. 2.2

Later, in Section 4, we will need the following statement. For h_F and d_F as in Lemma 2.2 a subset $X \subset VG_F$, as well as the cut δX in G_F, is called *tight* if $h_F(X) = d_F(X)$.

Statement 2.3. *Let $F \in B$, $|\mathcal{H}_F| \leq 2$, and let e be an edge in C_F with $h_F(e) > 0$. Then there exists a tight $X \subset VG_F$ such that $e \in \delta X$.*

Proof. Suppose that the statement is false for some e. Define $c'(e) := h_F(e) - 1/2$ and $c'(e') := h_F(e')$ for $e' \in EG_F - \{e\}$. Obviously, the problem (c', d_F) has a solution f'. One can see that the new f' together with a certain bunch B' contradict to the choice of f, B in (2.5), whence the result follows. •

Finally, for purposes of Section 4 we eliminate one more situation from further consideration.

Statement 2.4. *Let some $F \in B$ satisfy $h_F(e) = 1/2$ for all edges $e \in C_F$. Then $(G,U)^*$ has a half-integral solution.*

Proof. Consider the problems (h_F, d_F) and (c', d'), where $c'(e) := 1 - h_F(e)$ for $e \in EG_F$ and $d'(s,t) := 1 - d_F(s,t)$ for $\{s,t\} \in U$ (assuming that h_F and d_F are extended by zero to $EG - EG_F$ and $U - U_F$ respectively). It is easy to see that the value $2h_F(X) - 2d_F(X)$ as well as $2c'(X) - 2d'(X)$ is even for any $X \subseteq V$. Hence the problems $(2h_F, 2d_F)$ and $(2c', 2d')$ have integral solutions, and the result follows. •

3. PRIMITIVE METRICS

In this section we prove Theorem 4 and one more theorem that describes certain properties of the primitive $(G, \mathcal{H}, l)$ for which $l(e) = 4$ is achieved for some $e \in EG$. A face of a graph which is not a hole in it is called *intermediate*.

Consider a primitive $(G, \mathcal{H}, l)$ with $|\mathcal{H}| = 4$. Let G' be obtained from G by replacing each edge $e \in EG$ by $l(e)$ edges in series (if $l(e) = 0$ this means contraction of e); let $\mathcal{H}'$ be the corresponding set of holes for G'. It easy to see that $(G', \mathcal{H}', 1_{G'})$ is primitive, where $1_{G'}$ is the all-unit function on EG' (later on we say that a pair $(G'', \mathcal{H}'')$ is primitive if $(G'', \mathcal{H}'', 1_{G''})$ is primitive).

One may assume that $|\mathcal{H}'| = 4$ (as if $|\mathcal{H}'| \le 3$ then Theorem 4 immediately follows from Theorem 2; moreover, in this case $l(e) \le 2$ for each $e \in EG'$). In [Ka1] the following result was obtained, which, in fact, gives a strengthening of Theorem 3. Let ρ_X denote the cut metric on V induced by $X \subset V$ (i.e. $\rho_X(x, y)$ is 1 if $|X \cap \{x, y\}| = 1$ and 0 otherwise).

Theorem 3.1 [Ka1]**.** *Let $\widetilde{G}$ be a planar bipartite graph, $\widetilde{\mathcal{H}} \subseteq \mathcal{F}_{\widetilde{G}}$ and $|\widetilde{\mathcal{H}}| = 4$. Let $\{\delta X(1), \ldots, \delta X(k)\}$ be a maximal set of disjoint cuts in $\widetilde{G}$ so that*

$$\text{dist}^{\widetilde{G}}(s, t) = \text{dist}^{Q}(s', t') + \rho_{X(1)}(s, t) + \ldots + \rho_{X(k)}(s, t) \quad \text{for all } \{s, t\} \in W(\widetilde{\mathcal{H}}), \tag{3.1}$$

where Q is the graph obtained from $\widetilde{G}$ by contraction of the edges of $\delta X(1), \ldots, \delta X(k)$, and z' is the image of $z \in V\widetilde{G}$ in Q. Let $\mathcal{A}$ be the set of (non-void) faces in Q corresponding to those in $\widetilde{\mathcal{H}}$. Next, let $(H, \mathcal{B})$ be a pair obtained as a result of a maximal sequence of the following operations, each applying to the current pair $(Q', \mathcal{A}')$, starting from $(Q, \mathcal{A})$:

(i) *identifying two parallel edges in Q' bounding an intermediate face; or*

(ii) *identifying vertices $x, y \in VQ'$ belonging to the boundary of an intermediate face and such that: (a) $d(x, y)$ is even, and (b) for any $\{s, t\} \in W(\mathcal{A}')$, $d(s, t) \le \min\{d(s, x), d(s, y)\} + \min\{d(t, x), d(t, y)\}$; where d stands for $\text{dist}^{Q'}$.*

Then: either H consists of a unique vertex, or $|\mathcal{B}| = 3$ and H is a θ-graph (i.e. H is homeomorhpic to the graph of three parallel edges), or $|\mathcal{B}| = 4$ and H is a bipartite two- connected graph with $\mathcal{F}_H = \mathcal{B}$.

Now we begin to prove Theorem 4. Apply Theorem 3.1 to $(G', \mathcal{H}')$ as above.

From (3.1) and the primitivity of $(G', \mathcal{H}')$ it follows that $\{\delta X(1), \dots, \delta X(k)\}$ is empty, therefore, $G' = Q$ and $|\mathcal{F}_H| = 4$. Let σ' be the natural mapping of VG' to VH. It is a mapping onto VH. Furthermore, from (i),(ii) it follows that:

(3.2) σ' is naturally extended to a mapping of EG' to EH, and $\sigma'(EG') = EH$; for each $F' \in \mathcal{H}'$, σ' brings isomorphically $\mathrm{bd}(F')$ to the boundary of the corresponding face in H (denote this face as $\sigma'(F')$);

(3.3) an $s{-}t$ path P in G' with $\{s,t\} \in W(\mathcal{H}')$ is shortest if and only if the corresponding path $\sigma'(P)$ in H is shortest;

(3.4) one can add a set R of new edges in the interiors of some intermediate faces in G', preserving planarity, so that for each $V \in VH$ the subgraph $Z(v)$ in (VG', R) induced by $(\sigma')^{-1}(v)$ is a tree.

One can see that every $Z(v)$ connects the boundaries of as many holes in G' as the degree of v in H.

Denote by σ the restriction of σ' to VG. (3.2)-(3.4) imply that:

(3.5) σ can be extended to $EG \cup \mathcal{H}$ so that: (i) for each $e = xy \in EG$, $\sigma(e)$ is a path in H of length $l(e)$ connecting $\sigma(x)$ and $\sigma(y)$; and (ii) for $F \in \mathcal{H}$, $\sigma(F)$ is a face in H, σ states a one-to-one correspondence between $\mathcal{H}$ and $\mathcal{F}_H$, and σ brings the boundary of F to the boundary of $\sigma(F)$ (with preserving orientation clockwise);

(3.6) if P is an l-shortest path in G then the image by σ of P is a shortest path in H.

[A metric m on VG induced by (H, σ) such that H is planar and bipartite, $|\mathcal{F}_H| = 4$, and (3.4)-(3.6) hold (with l to be the restriction of m to EG) is called a 4f-*metric*; this notion will be used in Sections 4-6.]

We say that a subset $\emptyset \neq B \subset EH'$ *reduces* a planar bipartite graph H' if

$$\mathrm{dist}^{H'}(s,t) = \mathrm{dist}_B(s,t) + \mathrm{dist}_{EH'-B}(s,t) \tag{3.7}$$

holds for all $\{s,t\} \in W(\mathcal{F}_{H'})$, where for $A \subseteq EH'$, $\mathrm{dist}_A(s,t)$ stands for the distance between vertices s and t in the graph H' with length 1 for the edges in A and length 0 for the edges in $EH' - A$. We say that H' is *primitive* if $(H', \mathcal{F}_{H'}, 1_{H'})$ is primitive, in other words, there is no B reducing H'.

Obviously, the primitivity of $(G, \mathcal{H}, l)$ implies that for H as in Theorem 3.1. We need a statement on a necessary and sufficient condition of the primitivity of H' in case $|\mathcal{F}_{H'}| = 4$. We call a *regular dual circuit* in H' a minimal nonempty sequence

$$D = (F_0, e_1, F_1, \dots, e_k, F_k)$$

such that: (i) $F_0 = F_k$; (ii) F_{i-1} and F_i are different faces and e_i is a common edge in $\mathrm{bd}(F_{i-1})$ and $\mathrm{bd}(F_i)$; (iii) e_i and e_{i+1} are *opposite* edges in $\mathrm{bd}(F_i)$ (letting $e_{k+1} := e_1$).

A regular dual circuit will be considered up to reversing and/or cyclical shifting it. Therefore D is determined uniquely by any its edge, and the set of regular dual circuits give a partition of EH'. We say that a subset $B \subseteq EH'$ is *symmetric* if B is the union of the edge-sets of some regular dual circuits.

Next, a vertex in H' of degree $\neq 2$ is called *essential*. Let $\mathcal{R}(H')$ denote the set of non-trivial paths in H' (considered up to reversing) whose end vertices and only them are essential; these paths are called *elementary*.

Suppose that for some $P = v_0v_1 \dots v_k$ in $\mathcal{R}(H')$ one holds $k = |P| > \mathrm{dist}^{H'}(v_0, v_k)$ $:= d$. It is easy to see that the cut $\{v_0v_1, v_iv_{i+1}\}$, where $i := (k+d)/2$, reduces H'. Hence, the graph H in Theorem 3.1 satisfies

(3.8) $$|P| = \mathrm{dist}^H(s,t) \ \text{ for any } s-t \text{ path } \ P \in \mathcal{R}(H).$$

In particular, from (3.8) it follows that

(3.9) If $P, P' \in \mathcal{R}(H)$ connect the same pair of vertices then $|P| = |P'|$.

Lemma 3.2. *Let H be a two-connected bipartite planar graph with $|\mathcal{F}_H| = 4$, and let H satisfy (3.8). A set $\emptyset \neq B \subset EH$ reduces H if and only if B is symmetric.*

Hence, H is primitive if and only if all its edges belong to the same regular dual circuit.

Fig. 3.1

Remark 3.3. The statement of the lemma does not remain, in general, true for $|\mathcal{F}_H| = 5$, as shown by the graph in Fig 3.1 (here $\{e, e', u, u'\}$ is a symmetric set for which (3.7) is violated for some s, t.)

Proof of Lemma 3.2. Up to a homeomorphism of the sphere, H is of one of the types $H_1, \dots, H_4$ drawn in Fig. 3.2.

One can check that for case $|\mathcal{F}_H| = 4$ from (3.8) it follows that

(3.10) if x, y are opposite vertices in $\mathrm{bd}(F)$, $F \in \mathcal{F}_H$, then $\mathrm{dist}^H(x,y) = |\mathrm{bd}(F)|/2$, i.e. each path in $\mathrm{bd}(F)$ connecting x and y is shortest.

Let B reduce H, and $F \in \mathcal{F}_H$. Then, by (3.10), for any two opposite vertices x, y in $\mathrm{bd}(F)$ the paths in $\mathrm{bd}(F)$ connecting x and y contain the same number of edges in

B. Considering all such pairs $\{x,y\}$ in $\mathrm{bd}(F)$ we conclude that $e \in \mathrm{bd}(F)$ belongs to B if and only if the edge $e' \in \mathrm{bd}(F)$ opposite to e does. Hence, B is symmetric.

Fig. 3.2

Conversely, we prove that a symmetric B satisfies (3.7) for all $\{s,t\} \in W(\mathcal{F}_H)$. This is easy to check when $H \approx H_1$ (H is homeomorphic to H_1); in this case all paths connecting the pair of essential vertices have the same length, by (3.9).

Suppose that $H \approx H_2$. Let $x_1,\ldots,x_4$ be the essential vertices as indicated in Fig. 3.2b; denote by a_i the distance between x_i and x_{i+1} (letting $x_5 := x_1$). Let for definiteness $a_1 \ge a_3$. Consider two cases.

(i) $a_1 < a_2 + a_3 + a_4$. For $i = 2,4$ let P_i be the elementary path connecting x_i and x_{i+1}. One can see that there are edges $e \in P_2$ and $e' \in P_4$ that are opposite in the boundary of each of two corresponding faces. Clearly the set $E' := \{e,e'\}$ is symmetric and reduces H.

(ii) $a_1 = a_2 + a_3 + a_4$. Let $E(x)$ denote the set of edges in H incident to $x \in VH$. One can check that the set E' formed by the edges incident to x_1 or x_2 is symmetric and reduces H.

In both cases, the graph arising as a result of contraction of E' is homeomorphic to one of H_1, H_2, H_3. Therefore the result for $H \approx H_2$ will be implied by the proof below for case $H \approx H_3$. For cases $H \approx H_3$ and $H \approx H_4$ the lemma will follow from a slightly more general statement which will be also used later on (e.g., in the proof of Statement 3.6). We say that a graph K is a *proto-graph* of a planar graph H' if K arises from H' as a result of a sequence of the following operations: choose two elementary paths L, L' of the current graph H'' such that they have the same ends, form the boundary of some face and satisfy $|L| = |L'|$, and then remove one of L, L' (i.e. delete its edges and inner vertices from H'').

Statement 3.4. *Let H' be a two-connected bipartite planar graph satisfying (3.8), and let B be a symmetric set in EH'. Let $F \in \mathcal{F}_{H'}$, let s,t be two vertices in* $\mathrm{bd}(F)$, *let R_1, R_2 be the $s-t$ paths in* $\mathrm{bd}(F)$, *and let* $b := b(s,t) := \min\{|B \cap R_i| \,:\, i = 1,2\}$.

Suppose that there is a proto-graph K of H' such that: (i) K contains $\mathrm{bd}(F)$, *(ii) each essential vertex in K has degree three, (iii) the boundary of each face in K contains at most three essential vertices, and (iv) if the boundary of some face of K is formed by three elementary paths Z_1, Z_2, Z_3 then $|Z_1| + |Z_2| \geq |Z_3|$. Then $|B \cap P| \geq b$ holds for any $s-t$ path P in H'.*

This completes the proof of the lemma as follows. Consider vertices s, t in $\mathrm{bd}(F)$, $F \in \mathcal{F}_H$, and the $s-t$ paths R_1, R_2 in $\mathrm{bd}(F)$. It is easy to see that there is a proto-graph K of H satisfying the hypotheses in Statement 3.4. Let for definiteness $b(s,t) = |B \cap R_1|$. Since B is symmetric, $|R_1| \leq |R_2|$; moreover, $b' := |B' \cap R_1| \leq |B' \cap R_2|$, where $B' := EH - B$. Now applying Statement 3.4 to an arbitrary $s-t$ path P in H we conclude that $\mathrm{dist}_B(s,t) = b$ and $\mathrm{dist}_{B'}(s,t) = b'$, whence (3.7) (for $H' = H$) follows.

For a path $P = v_0 v_1 \ldots v_k$ and $0 \leq i \leq j \leq k$, $P(v_i v_j)$ denotes the part of P from v_i to v_j. If $Q = z_0 z_1 \ldots z_r$ is a path with $z_0 = v_k$ then $P \cdot Q$ denotes the concatenated path $v_0 v_1 \ldots v_k z_1 \ldots z_r$.

Proof of Statement 3.4. Let $H' = H_0, H_1, \ldots, H_k = K$ be the sequence of graphs such that H_{i+1} is formed from H_i by choosing paths L_i, L'_i as in the definition of K and then by removing L_i.

Consider the sequence $P = P_0, P_1, \ldots, P_k$, where P_{i+1} is an $s-t$ path (perhaps not simple) in H_{i+1} such that $P_{i+1} = P_i$ if L_i is not a part of P_i, and P_{i+1} is formed from P_i by replacing L_i by L'_i otherwise. Using the facts that $|L_i| = |L'_i|$ and that B is symmetric for H', by induction on i it is easy to show that:

(3.11) if for $F' \in \mathcal{F}_{H_i}$, $\mathrm{bd}(F')$ is formed by two elementary paths Q, Q' and $|Q| \leq |Q'|$ then $|B \cap Q| \leq |B \cap Q'|$;

(3.12) if for $F' \in \mathcal{F}_{H_i}$, $\mathrm{bd}(F')$ is formed by three elementary paths Q, Q', Q'' and $|Q| \leq |Q'| + |Q''|$ then $|B \cap Q| \leq |B \cap Q'| + |B \cap Q''|$.

From (3.11) it follows that $|B \cap P_i| = |B \cap P|$ for all i. Next, the conditions (ii)-(iii) on K imply that K is either a θ-graph or a circuit or $K \approx K_4$. Let P' be a simple $s-t$ path in P_k; then $|B \cap P'| \leq |B \cap P_k|$. If P' belongs to $\mathrm{bd}(F)$ then the statement is obvious. If K is a θ-graph, the statemets follows from (3.11). Otherwise $K \approx K_4$ and P' is of form $Q_1 \cdot Q_2 \cdot Q_3 \cdot Q_4$, where Q_2 is an $x-y$ path, Q_3 is an $y-z$ path, the vertices x, z belong to $\mathrm{bd}(F)$, and y is the essential vertex not in $\mathrm{bd}(F)$. Then the statement follows from the condition (iv) and (3.12). ••

Remark 3.5. Statement 3.4 proves Lemma 3.2 for all cases of H's, including $H \approx H_1$ or H_2. Note that for case $H \approx H_2$ arguments in the proof of the lemma show that H is not primitive. Also if $H \approx H_1$ then obviously H is not primitive unless all paths connecting the essential vertices are of length 1 (in this case $l(e') \leq 1$ for all $e' \in EG$). These facts will be used in what follows.

Return to consideration $(G, \mathcal{H}, l), G', H, \sigma, \sigma'$ as above. Suppose that there is $e = xy \in EG$ with $l(e) \geq 4$. The edge e corresponds to a path L_e in H connecting the vertices $b_0 := \sigma(x)$ and $b_k := \sigma(y)$ $(k = l(e))$. Note that e belongs to an l-shortest $s-t$ path in G with $\{s,t\} \in W(\mathcal{H})$ (for otherwise the function l' on EG, defined by $l'(e) := 2$ and $l'(e') := 0$ for all $e' \in EG - \{e\}$, obviously, satisfies (1.7) (i.e. l' $\mathcal{H}$-reduces l)). Therefore, by (3.6),

(3.13) L_e is a part of some shortest path $\widetilde{L}$ in H connecting vertices in the boundary of some its face.

Now we construct a graph $\overline{H}$ with $|\mathcal{F}_{\overline{H}}| = 5$ as follows. By the definition of G', e corresponds to a path $L' = v_0 v_1 \dots v_k$ in G' consisting of $k = l(e)$ edges. Split G' along L'. More precisely, we form the graph G'' containing simple paths $L_1 = u_0 u_1 \dots u_k$ and $L_2 = w_0 w_1 \dots w_k$ so that: (i) $u_0 = u_k$ and $w_0 = w_k$; (ii) L_1 and L_2 form the boundary of a face F' in G''; (iii) G' is obtained from G'' by identifying u_i and w_i (which results in the vertex v_i), $i = 0, \dots, k$, and by identifying the edges $u_i u_{i+1}$ and $w_i w_{i+1}$, $i = 0, \dots, k-1$ (so F' disappears).

Such an operation naturally splits each tree $Z(v_i)$ (defined in (3.4)), $i = 1, \dots, k-1$, into two trees: the tree Z'_i containing u_i and the tree Z''_i containing w_i. Now $\overline{H}$ is obtained from G'' by shrinking each $Z(v)$ for $v \in VG' - \{v_1, \dots, v_{k-1}\}$, by shrinking each Z'_i and each Z''_i and then by identifying parallel edges forming the boundaries of intermediate faces.

One can see that $\overline{H}$ can be obtained directly from H by "splitting" the path L_e in two "parallel" paths $\overline{L}_1$ ("right") and $\overline{L}_2$ ("left") that form the boundary of a new intermediate face; denote it by F (see Fig. 3.3 for illustration). Let $\overline{\mathcal{H}} := \mathcal{F}_{\overline{H}} - \{F\}$, and let $L_e = b_0 b_1 \dots b_k$, $\overline{L}_1 = x_0 x_1 \dots x_k$ and $\overline{L}_2 = y_0 y_1 \dots y_k$, where $b_0 = x_0 = y_0$ and $b_k = x_k = y_k$.

Fig. 3.3

It is not difficult to show that the primitiwity of $(G, \mathcal{H}, l)$ implies the primitivity of $(\overline{H}, \overline{\mathcal{H}})$. In essense, we show later on that if $l(e) \geq 5$ then $(\overline{H}, \overline{\mathcal{H}})$ is not primitive.

Consider the regular dual circuit $D = (F_0, e_1, F_1, \dots, e_r, F_r)$ in H (it contains all edges of H, by Lemma 3.2). We say that an edge $u = b_i b_{i+1} \in L_e$ is *right* (*left*) if $u = e_j$ and F_{j-1} lies on the right (left) from u (regarding u as oriented from b_i to b_{i+1}).

In the further proof we shall consider a certain pair u, u' of right (or left) edges in L_e and construct an auxiliary graph $H' = H(u, u')$ and a dual circuit $D' = D(u, u')$ (not necessarily regular) in it, as follows. Let for definiteness $u = b_i b_{i+1}$, $u' = b_j b_{j+1}$, $i < j$, and both u, u' are right. Then H' arises from $\overline{H}$ by identifying the vertices x_d and y_d for each $d \le i$ and $d \ge j+1$, and then by identifying appeared parallel edges. Denote by $\mathcal{A} = \mathcal{A}(u, u')$ the set of faces in H' corresponding to the faces of H. For definiteness one may assume that $u = e_p$, $u' = e_q$ and $p < q$. Then D' is obtained from the part $(F_p, e_{p+1}, \ldots, e_{q-1}, F_q, e_q)$ of D by:

(i) replacing e_q by the sequence $(x_j x_{j+1}, F, y_i y_{i+1}, F_p)$;

(ii) replacing e_s by the sequence $(x_t x_{t+1}, F, y_t y_{t+1})$ if $e_s = b_t b_{t+1}$, $i < t < j$, and e_s is right;

(iii) replacing e_s by the sequence $(y_t y_{t+1}, F, x_t x_{t+1})$ if $e_s = b_t b_{t+1}$, $i < t < j$, and e_s is left.

Let $B = B(u, u')$ be the set of edges in D', L'_1 be the part $\overline{L}_1(x_i, x_j)$ of $\overline{L}_1$ and $L'_2 := \overline{L}_2(y_i, y_j)$. One can see that

(3.14) $\emptyset \ne B \subset EH'$, $|B \cap L'_1| = |B \cap L'_2|$, and B is symmetric for every face $F' \in \mathcal{A}$ (i.e. $B' := B \cap \mathrm{bd}(F')$ is the union of some pairs of opposite edges).

We call a *segment* a maximal non-trivial part L in L_e whose inner vertices are of degree 2 (in H). Represent L_e as $L_1 \cdot L_2 \cdot \ldots \cdot L_d$, where each L_i is a segment.

Statement 3.6. *Each segment L_i contains at most one right edge and at most one left edge.*

Proof. Suppose that L_i contains edges $u = b_i b_{i+1}$ and $u' = b_j b_{j+1}$ that are both right (or left). One may assume that u, u' are chosen so that $|i - j| \le 2$. Then, obviously, B is symmetric for all faces, inclyding F. Consider the graph $H' = H(u, u')$ and the sets $B = B(u, u')$ and $\mathcal{A} = \mathcal{A}(u, u')$. One can see that for each $s, t \in \mathrm{bd}(F')$, $F' \in \mathcal{A}$, there is a proto-graph of H' which satisfies the hypotheses in Statement 3.4. Now from (3.14) and Statement 3.4 it follows that (3.7) holds for all $\{s, t\} \in W(\mathcal{A})$, i.e. $(H', \mathcal{A})$ is not primitive. Define l' on EG by $l'(e) := |L \cap B|$ and $l'(e') := |L_{e'} \cap B|$ for $e' \in EG - \{e\}$, where L is a path in H' corresponding to e, and $L_{e'}$ is the path in H' corresponding to e'. Then $l' \le l$ and $l' \ne 0, l$. Now from (3.6) and the constructions of $H, \overline{H}, H'$ one can deduce that l' satisfies (1.7). Hence, $(G, \mathcal{H}, l)$ is not primitive; a contradiction. •

From Statement 3.6 it follows that

(3.15) for $i = 1, \ldots, d$, $|L_i| \le 2$, and if $|L_i| = 2$ then L_i contains one right edge and one left edge.

Now suppose that $l(e) \geq 4$. In view of Remark 3.5, it suffices to consider only cases $H \approx H_3$ and $H \approx H_4$.

Let $H \approx H_3$. Let x, y, z and $P_1, \ldots, P_4, L$ be the vertices and the paths as indicated in Fig. 3.2c, and let $a_i := |P_i|$. Then $a_1 = a_2$, $a_3 = a_4$ and $|L| \geq |a_1 - a_3|$. Consider the vertex v in L such that $a_1 + |L'| = a_3 + |L''|$, where L' (L'') is the part of L between y and v (resp., z and v). One may assume that each P_i has the first vertex at x and that the first edge in P_1 is right (with respect to D). Since all simple paths in H from x to v have the same length, one can see that

(3.16) all edges in every simple $x - v$ path are right.

Now using (3.15),(3.16) and the fact that L_e is contained in a shortest $s - t$ path for some $\{s, t\} \in W(\mathcal{F}_H)$, a straightforward examination shows that the only possible case is:

(3.17) $a_1 = a_3 \geq 3$, $|L| = 2$, $l(e) = 4$, $\{y, z\} = \{b_1, b_3\}$, and b_0, b_4 belong to either $P_1 \cup P_4$ or $P_2 \cup P_3$;

Fig. 3.4

see Fig. 3.4a. Thus, for $H \approx H_3$, $l(e) \geq 5$ is impossible. Moreover, one can see that

(3.18) if $H \approx H_3$ and $l(e) = 4$ then $(G, \mathcal{H}, l)$ is not primitive.

Indeed, for $u = b_0b_1$ and $u' = b_1b_2$ consider the graph $H' = H(u, u')$ and the set $B = B(u, u')$, defined as above. Clearly B is a proper subset of EH' and B is symmetric. Using arguments as in the proof of Statement 3.4, one can deduce from (3.17) that (3.7) holds for all $\{s, t\} \in W(\mathcal{F}_{H'})$. Thus, H' is not primitive, whence $(G, \mathcal{H}, l)$ is not primitive as well.

Now we consider case $H \approx H_4$. Let x_i and $F(i)$ ($i = 0, \ldots, 3$) be the essential vertices and the faces in H, as shown in Fig. 3.2d. Denote by P_{ij} the elementary path in H from x_i to x_j.

Statement 3.7. *Suppose that the paths $P := P_{ij} \cdot P_{jp}$ and $P' := P_{ij} \cdot P_{jq}$ are shortest for some distinct i, j, p, q. Then:*

(i) *there is a vertex* $v \in P_{pq}$ *such that all simple paths from* x_i *to* v *have the same number of edges;*

(ii) *the edges of all paths from* x_i *to* v *are simultaneously right or simultaneously left (with respect to* D*).*

Proof. By (3.8), $|P_{ip}| = |P|$, $|P_{iq}| = |P'|$ and $|P_{pq}| \geq ||P| - |P'||$. This implies (i). (ii) easily follows from (i). •

Combining (3.15) and Statement 3.7 with the fact that L_e is a part of a shortest $s - t$ path with $\{s, t\} \in W(\mathcal{F}_H)$, one can deduce that

(3.19) if $P_{ij} \cdot P_{jp}$ and $P_{ij} \cdot P_{jq}$ are shortest for some distinct i, j, p, q then the only possible case is: $l(e) = 4$, $|P_{pq}| = 2$, $|P_{ip}| = |P_{iq}| \geq 3$, $\{b_1, b_3\} = \{x_p, x_q\}$, and b_0, b_4 belong to either $P_{ip} \cup P_{iq}$ or $P_{jp} \cup P_{jq}$;

(see Fig. 3.4b). Hence, in case as in Statement 3.7, $l(e) \geq 5$ is impossible. Moreover, in this case

(3.20) $|P_{jp}| = |P_{jq}| \geq 2$, and therefore, P_{pq} coincides with $b_1 b_2 b_3$.

Indeed, if $|P_{jp}| = |P_{jq}| = 1$ then the cut $\{u, u', u''\}$ reduces H, where u is the first edge in P_{ip}, u' is the edge in P_{jp} and u'' is the second edge in P_{pq}.

In what follows we assume that no i, j, p, q as in Statement 3.7 exist.

Statement 3.8. *There are no three essential consecutive vertices* b_q, b_{q+1}, b_{q+2} *in* L_e*.*

Proof. Suppose that it is not so, and let for definiteness $b_q = x_0$, $b_{q+1} = x_1$, $b_{q+2} = x_2$. Put $a_{ij} := |P_{ij}|$. Since $a_{01} = 1$, $|a_{03} - a_{13}| = 1$ (as H is bipartite). We know that $P_{01} \cdot P_{12}$ is shortest, so $P_{01} \cdot P_{13}$ cannot be shortest, whence the case $a_{03} > a_{13}$ is impossible. Hence, $a_{13} = a_{03} + 1$, and the path $P_{10} \cdot P_{03}$ is shortest. Similarly, $a_{13} = a_{23} + 1$, and the path $P_{12} \cdot P_{23}$ is shortest. In addition, obviously $|P_{02}| = 2$. These facts easily imply that if $|P_{03}| > 1$ then $\{u_0, u_1, u_2, w_0, w_1, w_2, z_1, z_2\}$ is a proper symmetric set in H, where u_0, u_1, u_2 are the first edges in P_{30}, P_{31}, P_{32}, respectively, $w_0 \in P_{01}$, $w_2 \in P_{21}$, $z_1, z_2 \in P_{02}$ and w_1 is the last edge in P_{30}; a contradiction. And if $|P_{03}| = 1$ then L_e cannot be shortest (even for $|L_e| \geq 3$). •

In view of $l(e) \geq 4$, from Statements 3.6 and 3.8 it follows that there are two consecutive segments L_i, L_{i+1} in L_e such that $|L_i| + |L_{i+1}| \geq 3$. Then there are edges $u = b_q b_{q+1}$ and $u' = b_{q'} b_{q'+1}$ ($q < q'$) in $L_i \cdot L_{i+1}$ such that both u, u' are right or both u, u' are left. Let u, u' be chosen so that $q' - q$ is as small as possible. One may assume that both u, u' are right, $u \in P_{01}$, $u' \in P_{12}$ and $b_{q+1} = x_1$. Then either $q' = q + 1$ or $q' = q + 2$ (in the latter case the edge $b_{q+1} b_{q'}$ is left). Put $y := b_q$, $v := b_{q'}$ and $z := b_{q'+1}$ (so $v = x_1$ if $q' = q + 1$). The following statement is key in the proof.

Statement 3.9. *At least one of the following paths is shortest: (i)* $\overline{P} := P_{10} \cdot P_{02}$*; (ii)* $\overline{P} := P_{02} \cdot \widetilde{P}$*, where* $\widetilde{P}$ *is the part of* P_{21} *from* x_2 *to* v*.*

Proof. Form the graph $H' = H(u, u')$ and the set $B = B(u, u')$ as it was explained before Statement 3.6. One may assume that H' is obtained from H by adding a vertex v' and edges $yv', v'z$ in case $q' = q+1$ or by adding vertices v', x' and edges $yv', v'x', x'z$ in case $q' = q + 2$, see Fig 3.5. In case $q' = q + 1$ put $x' := v'$.

Denote by Q the path yvz (resp., yx_1vz), and by Q' the path $yv'z$ (resp., $yv'x'z$); let F be the face in H' bounded by Q and Q', and let $\mathcal{A} := \mathcal{F}_{H'} - \{F\}$.

Fig. 3.5

The minimal choice of $q' - q$ implies that B is symmetric (for all faces). Furthermore, B contains exactly one pair among $\{u, \overline{u}\}$ and $\{u', \overline{u}'\}$, where $\overline{u} := zx'$, $\overline{u}' := yv'$; one may assume that B contains $\{u', \overline{u}'\}$ (otherwise consider $\overline{B} := EH' - B$ instead of B). In fact, we show that if the statement is false then B reduces $(H', \mathcal{A})$ (i.e. (3.7) holds for all $\{s, t\} \in W(\mathcal{A})$), which will imply that $(G, \mathcal{H}, l)$ is not primitive.

Suppose that for some $I' \in \mathcal{A}$ and $s, t \in \mathrm{bd}(I')$ there exists an $s - t$ path P in H' such that $|B \cap P| < |B \cap R_i|$ or $|\overline{B} \cap P| < |\overline{B} \cap R_i|$ for $i = 1, 2$, where R_1, R_2 are the $s - t$ paths in $\mathrm{bd}(I')$. Let, in addition, P be chosen so that: (i) $|P|$ is minimum; (ii) the number of edges in $P \cap \mathrm{bd}(I')$ is maximum, subject to (i); and (iii) P separates the minimum number of pairs of faces in $\mathcal{F}_{H'} - \{I'\}$, subject to (i),(ii).

From (i) it easily follows that P is simple, its ends are essential and no inner vertex of P belongs to $\mathrm{bd}(I')$. Next, there is no face F' such that $\mathrm{bd}(F') = M \cdot R$, where R is a part of P, and either $|M| < |R|$, or $|M| = |R|$ and for the path P' obtained from P by replacing the part R by M the value as in (ii) becomes more, or the values as in (i)-(ii) remain the same but the value in (iii) decreases. For otherwise we have $|B \cap P'| \le |B \cap P|$ and $|\overline{B} \cap P'| \le |\overline{B} \cap P|$ (in view of $|M| \le |R|$ and the symmetry of B), coming to a contradiction with condition (i),(ii) or (iii).

Let the faces in $\mathcal{A}$ be denoted as in Fig. 3.5. It is easy to see that in case $I' = F(i)$ for $i \in \{0, 1\}$, from said above it follows that P is contained in $\mathrm{bd}(F(i))$; a contradiction.

Now suppose that $I' = F(2)$. Notice that if P contains Q' then replacing it by Q we would get a path P' which contradicts to condition (ii) for P. Now from said above we can conclude that the only possible case is: $\{s,t\} = \{x_0, x_1\}$ and P (or the opposite path) is $P_{02} \cdot P_{21}$. Let L be the simple $z - x_1$ paths in bd(F) that contains zv, and let L' be the other $z - x_1$ path in bd(F). Denote by P' the path obtained from P by replacing the part L by L'.

We have $|L'| = |L| + 2$, $|B \cap L'| = |B \cap L|$ and $|\overline{B} \cap L'| = |\overline{B} \cap L| + 2$ (as $u, \overline{u}$ belong to $\overline{B}$). Hence, $|P'| = |P| + 2$, $|B \cap P'| = |B \cap P|$ and $|\overline{B} \cap P'| = |\overline{B} \cap P| + 2$. On the other hand, in view of the symmetry of B for $F(1)$ and the fact that $|P| \geq |P_{01}|$, we have $|P'| \geq |P_{01}| + 2$, $|B \cap P'| \geq |B \cap P_{01}|$ and $|\overline{B} \cap P'| \geq |\overline{B} \cap P_{01}|$. Hence, $|B \cap P| \geq |B \cap P_{01}|$, therefore, by supposition on P, one must be $|\overline{B} \cap P| < |\overline{B} \cap P_{01}|$. Then, obviously, $|\overline{B} \cap P_{01}| = |\overline{B} \cap P'|$. This means that the edge in bd$(F(1))$ opposite to $\overline{u} = zx'$ belongs to P_{01}. Then the length of the $x_0 - x'$ path in bd$(F(1))$ that contains z is at most $|\text{bd}(F(1)|/2$. This implies that $|P_{02} \cdot \widetilde{P}| \leq |P_{01}| + 1$, whence it easily follows that $P_{02} \cdot \widetilde{P}$ is shortest.

Finally, let $I' = F(3)$. Then $P = P_{10} \cdot P_{02}$. Repeating arguments as in the previous case we conclude that $|\overline{B} \cap P| \geq |\overline{B} \cap P_{12}|$, and that $|B \cap P| < |B \cap P_{12}|$ is possible only if $P_{10} \cdot P_{02}$ is shortest. •

In particular, Statement 3.9 enables us to establish the following fact (which, together with (3.18), will be important for the proof of Theorem 3.12):

(3.21) if $H \approx H_4$, the paths $P_{ij} \cdot P_{jp}$ and $P_{ij} \cdot P_{jq}$ are shortest for some distinct i, j, p, q and $l(e) = 4$, then $(G, \mathcal{H}, l)$ is not primitive.

Indeed, by (3.19), the edges $u = b_0 b_1$ and $u' = b_1 b_2$ are either both right or both left. In view of (3.19)-(3.20), one can apply Statement 3.9 to $H(u, u')$ and $B(u, u')$. Then at least one of the paths $P_{sp} \cdot P_{pq}$, $P_{sq} \cdot P_{qp}$ or $P_{ps} \cdot P_{sq}$ is shortest, where $r := i$ if b_0, b_4 belong to $P_{ip} \cup P_{iq}$, and $r := j$ if they belong to $P_{jp} \cup P_{jq}$. But this is impossible since $|P_{rp}| = |P_{rq}|$ and $|P_{ps}| + |P_{sq}| > |P_{pq}| = 2$.

The following assertion strengthens Statement 3.8.

Statement 3.10. *There do not exist two consecutive segments $L_i = b_p \dots b_q$ and $L_{i+1} = b_q \dots b_r$ in L_e so that the vertices p and r are essential (in H).*

Proof. This immediately follows from Statement 3.8 if $q - p = r - q = 1$. Let for definiteness $b_p = x_0$, $b_q = x_1$, $b_r = x_2$ and $r - q = 2$. Consider the edges $u := b_{q-1} b_q$ (in L_i) and $u' := b_{q'} b_{q'+1}$ (in L_{i+1}) such that u, u' are both right or both left. By Statement 3.9, one of the paths $P_{10} \cdot P_{02}$ or $P_{02} \cdot \widetilde{P}$ is shortest, where $\widetilde{P}$ is the part of P_{21} from $b_r = x_2$ to $b_{q'}$. But this is impossible because the path $b_p \dots b_r = P_{01} \cdot P_{12}$ is also shortest. •

This statement implies that

(3.22) the number d of segments in L_e is two or three, and L_e contains at most two essential vertices.

Put $a_i := |L_i|$, $i = 1, \dots, d$. Consider case $d = 3$. Let $L_1 = b_0 \dots b_p$, $L_2 = b_p \dots b_q$, $L_3 = b_q \dots b_k$. Without loss of generality one may assume that $a_1 + a_2 \geq 3$, $L_1 \subseteq P_{21}$, $L_2 = P_{10}$; then $b_p = x_1$, $b_q = x_0$, and $L_3 \subseteq P_{0,i}$ for $i \in \{2, 3\}$. Consider two cases.

Case 1. $L_3 \subseteq P_{03}$, see Fig. 3.6a. Consider a shortest $s - t$ path L containing L_e, where $\{s, t\} \in \mathrm{bd}(I')$, $I' \in \mathcal{F}_H$. If L passes through x_2 then we come to a contradiction using arguments as in the proof of Statement 3.10. Hence, $I' = F(3)$, and L contains $P_{10} \cdot P_{03}$. Then, in view of (3.21), the path $P_{10} \cdot P_{02}$ is not shortest. Consider a path $\overline{P}$ as in Statement 3.9 for corresponding u, u' in $L_1 \cup L_2$. Then: (i) either $\overline{P} = P_{02} \cdot P_{21}$, or (ii) $\overline{P} = P_{02} \cdot \widetilde{P}$, where $\widetilde{P}$ is the part of P_{21} from x_2 to b_{p-1} (and then $a_1 = 2$), or (iii) $\overline{P} = \widetilde{P} \cdot P_{02}$, where $\widetilde{P}$ is the part of P_{10} from b_{p+1} to x_0 (and then $a_2 = 2$). As $L_1 \cdot L_2$ is shortest, cases (i),(ii) are impossible. In case (iii), consider the pair L_2, L_3. Then $a_2 + a_3 \geq 3$, and now taking into account the fact that $P_{10} \cdot P_{03}$ is shortest, we get a contradiction using arguments as in the proof of Statement 3.10.

Fig. 3.6

Case 2. $L_3 \subseteq P_{02}$; see Fig. 3.6b. Aplying Statement 3.9 to L_1, L_2 and to L_2, L_3 (if $a_2 + a_3 \geq 3$), and using the fact that L_e is shortest one can show that only two cases are possible:

(3.23) $a_1 = a_3 = 1$, $a_2 = 2$ and the paths $P \cdot P_{02}$ and $P' \cdot P_{12}$ are shortest, where $P = b_2 b_3$ and $P' = b_2 b_1$; or

(3.24) $a_1 = 2$, $a_2 = a_3 = 1$, the path $P_{10} \cdot P_{02}$ is shortest, and $|P_{02}| \geq 3$.

Finally, consider case $d = 2$. Then $a_i \leq 2$ implies $a_1 = a_2 = 2$. Let for definiteness, $L_1 = b_0 b_1 b_2 \subseteq P_{21}$, $L_2 = b_2 b_3 b_4 \subseteq P_{10}$, and the vertex b_0 is not essential; see Fig. 3.6c. From Statement 3.9 it follows that

(3.25) at least one of the following is true: (i) $P_{10} \cdot P_{02}$ is shortest; (ii) $P_{12} \cdot P_{20}$ is shortest;

(iii) the paths $P \cdot P_{02}$ and $P' \cdot P_{20}$ are shortest, where P is the part of P_{10} from b_3 to x_0 and P' is the part of P_{12} from b_1 to x_2.

This completes the proof of Theorem 4. • • •

In fact, from the above proof one can obtain a stronger, in a sense, result, as follows. For a planar graph G' and a face F in it, a path P in G' with both ends in $\mathrm{bd}(F)$ is called an *F-path*.

Theorem 3.11. *Let G' be a planar graph, $\mathcal{H}' \subseteq \mathcal{F}_{G'}$ and $|\mathcal{H}'| = 4$. Let m be a 4f-metric on VG' induced by (H, σ) such that $(G', \mathcal{H}', m|_{EG'})$ is primitive and $m(e) = 4$ for some $e \in EG'$. Then:*

(i) *$H \approx K_4$, the image by σ of e is a shortest path $L_e = b_0b_1b_2b_3b_4$ in H which belongs to the boundary of a (unique) face $\widetilde{J}$ in H;*

(ii) *each shortest path in H connecting two vertices in $\mathrm{bd}(\widetilde{J}) - \{b_1, b_2, b_3\}$ lies in $\mathrm{bd}(\widetilde{J})$;*

(iii) *suppose that there is $\widetilde{I} \in \mathcal{F}_H - \{\widetilde{J}\}$ such that some shortest $\widetilde{I}$-path in H contains b_0 and some shortest $\widetilde{I}$-path contains b_4; then:*

  (a) no shortest $\widetilde{I}$-path contains both b_0 and b_4;

  (b) if $b \in \{b_0, b_4\}$ is not in $\mathrm{bd}(\widetilde{I})$ and L is a shortest $\widetilde{I}$-path containing b then L separates $\widetilde{J}$ from $\widetilde{K}$ and $\widetilde{O}$, where $\mathcal{F}_H = \{\widetilde{I}, \widetilde{J}, \widetilde{K}, \widetilde{O}\}$;

  (c) no shortest $\widetilde{I}$-path contains an edge common for $\mathrm{bd}(\widetilde{K})$ and $\mathrm{bd}(\widetilde{O})$.

(We say that an I'-path L separates J' and K' if they lie in different components of the space $\mathbb{R}^2 - (I' \cup L)$.)

Proof. (i) has been already proved (see (3.21),(3.18) and arguments in the proof of Lemma 3.2 showing that H is homeomorpic to neither H_1 nor H_2). (a) in (iii) can be checked using (3.8)(3.21),(3.23)- (3.25), Statement 3.9 and the fact that L_e is shortest (a check-up is easy and we leave it to the reader). Let us prove (ii). By (3.21), there are no distinct i, j, p, q such that the paths $P_{ij} \cdot P_{jp}$ and $P_{ij} \cdot P_{jq}$ are shortest. We use notations as in Fig. 3.2d and assume that the case as in one of (3.23)-(3.25) takes place.

Suppose that L is a shortest $s - t$ path in H such that $s, t \in \mathrm{bd}(\widetilde{J}) - \{b_1, b_2, b_3\}$ and that L has a vertex not in $\mathrm{bd}(\widetilde{J})$. Then L passes through x_3. Three cases are possible.

Case 1. L contains $P_{03} \cdot P_{31} \cdot L_1$ (here $L_1 \subset P_{12}$ is the first segment in L_e). Then no shortest path can pass through x_0, x_2, b_1 or through x_1, x_0, x_2 (in these orders). Hence, cases (3.24) and (3.25) are impossible. In case (3.23), $|P_{01}| = 2$ implies $|P_{03}| = |P_{31}| = 1$. Since $P_{23} \cdot P_{31}$ cannot be shortest (by (3.21)), we have $|P_{23}| = |P_{21}| + 1$, hence the paths P_{23}, $P_{21} \cdot P_{13}$ and $P_{20} \cdot P_{03}$ are shortest. Then the edges in P_{01}, P_{03}, P_{13} together with the first edges in P_{20}, P_{21}, P_{23} and the last edge in P_{23} form a proper symmetric subset in EH; a contradiction.

Case 2. L contains $P_{23} \cdot P_{31} \cdot L_2$ (L_2 is the second segment in L_e). By arguments as in the previous case, $d = 3$. Then $x_0 = b_3$, whence L connects vertices in $P_{02} - \{x_0\}$. So P_{02} is not shortest; a contradiction with (3.8).

Case 3. L contains $P_{03} \cdot P_{32}$. Then $d = 2$ (otherwise L would connect vertices in $P_{02} - \{x_0\}$). If $P \cdot P_{02}$ and $P' \cdot P_{20}$ are shortest (see (3.25)), then from $|P| + |P_{02}| \leq |P_{12}| + 1$ and $|P'| + |P_{20}| \leq |P_{10}| + 1$ it follows that $|P_{02}| \leq 2$. Hence, without loss of generality it suffices to consider two cases: (i) $|P_{10}| = |P_{12}| =: k$ and $|P_{02}| = 2$; and (ii) $P_{10} \cdot P_{02}$ is shortest.

In case (i), the fact that $P_{03} \cdot P_{32}$ is shortest implies that $|P_{03}| = |P_{32}| = 1$. Then $|P_{13}| = k + 1$ (otherwise $P_{03} \cdot P_{31}$ and $P_{23} \cdot P_{31}$ would be shortest, contrary to (3.21)). By arguments as in Case 1, there is a proper symmetric subset in EH; a contradiction.

In case (ii), considering L and $P_{10} \cdot P_{02}$ we get that the path $P_{10} \cdot P_{03} \cdot P_{32}$ is shortest. Hence, we have two shortest paths $P_{10} \cdot P_{03}$ and $P_{10} \cdot P_{02}$; a contradiction with (3.21).

Thus, (ii) is true. Now we prove (iii)(b). Let L and b be as in (iii)(b). First of all we observe that L does not contain $P_{01} \cdot P_{12}$ (otherwise none of the paths pointed out in (3.23)-(3.25) could be shortest). Suppose that L does not separate $\widetilde{J}$ from $\widetilde{K}$ or from $\widetilde{O}$. We first eliminate some cases for L.

Case 4. L contains $P_{12} \cdot P_{23}$. Then $P_{12} \cdot P_{20}$ is not shortest (by (3.21)). If $P_{10} \cdot P_{02}$ is shortest, we get a contradiction as in Case 3(ii). Thus, two cases are possible.

(i) The case as in (3.23). If $\widetilde{I} = F(0)$ then L must contain P_{01}; a contradiction as in Case 3(ii). If $\widetilde{I} = F(2)$ then the path passing through $b_4 \in P_{02}$ contains either $P_{02} \cdot P_{21}$ or $P_{02} \cdot P_{23}$; a contradiction with (3.21).

(ii) $d = 2$, the paths $P \cdot P_{02}$ and $P' \cdot P_{20}$ are shortest and $|P_{10}| = |P_{12}| =: k$ (see (3.25)). Then $|P_{02}| = 2$. Put $r := |P_{23}|$; then $|P_{13}| = k + r$. Considering P_{03}, P_{02} and P_{23} we observe that $||P_{03}| - r|$ is an even ≤ 2 (by (3.8)), and considering P_{03}, P_{01}, P_{13} we see that $|P_{03}| \geq r$ (by (3.8) for P_{13}). If $|P_{03}| = r + 2$ then $P_{02} \cdot P_{23}$ is shortest; a contradiction with (3.21). If $|P_{03}| = r$ then the edges in P_{02} together with the first edges in $P_{10}, P_{12}, P_{30}, P_{32}$ and the first first and last edges in P_{13} form a proper symmetric set; a contradiction.

Case 5. L contains $P_{10} \cdot P_{03}$. If $d = 2$, the proof as in Case 4. Let $d = 3$. Since L passes through $b \notin \mathrm{bd}(\widetilde{I})$, it must contain P_{12}; a contradiction as in Case 3(ii).

Case 6. L contains $P_{20} \cdot P_{03}$. This case arises only when L passes through b_4 (since if L passes through $b_0 \notin \mathrm{bd}(\widetilde{I})$ then L contains P_{12}; a contradiction as in Case 3(ii)). From (3.23)- (3.25) it follows that $|P_{01}| \leq 2$. In addition, $P_{10} \cdot P_{02}$ is not shortest (by (3.21)).

(i) The case as in (3.23). Then our case is similar to Case 4(i).

(ii) The case as in (3.25). From the facts that $|P_{01}| = 2$ (since $b_4 = x_0$) and that L_e is shortest it follows that the only possible case is when $P_{10} \cdot P_{12}$ is shortest (in view of (3.25)). A contradiction with (3.21).

Case 7. L contains $P_{02} \cdot P_{23}$. Clearly only $\widetilde{I} = F(2)$ is possible. Then there is a shortest path L' containing $P_{12} \cdot P_{23}$ or $P_{12} \cdot P_{20}$ (as b_0 is neither in $\mathrm{bd}(\widetilde{I})$ nor in L); a contradiction with (3.21).

Thus, L contains either $P_{02} \cdot P_{21}$ (and then $\widetilde{I} = F(2)$) or $P_{10} \cdot P_{02}$ (and then $\widetilde{I} = F(3)$). This proves (b) in (iii). Finally, if $P_{02} \cdot P_{21} \subseteq L$ then, by (3.21), neither $P_{02} \cdot P_{23}$ nor $P_{12} \cdot P_{23}$ are shortest. Hence, no shortest $F(2)$-path contains an edge in $\mathrm{bd}(F(3)) \cap \mathrm{bd}(F(0))$. Similarly, if $P_{10} \cdot P_{02} \subseteq L$ then no shortest $F(3)$-path has an edge in $\mathrm{bd}(F(2)) \cap \mathrm{bd}(F(0))$.

The proof of Theorem 3.11 is complete. •

Theorems 3,4 and 3.11 give the following consequence.

Corollary 3.12. *Let $|\mathcal{H}| = 4$, $c : EG \to I\!R_+$, $d : U \to I\!R_+$, and let the problem (c, d) have no solution. Suppose that $c(m') \geq d(m')$ holds for all cut- and 2,3-metrics m' on VG. Then there exists a 4f-metric m on VG induced by (H, σ) such that $c(m) < d(m)$ holds and $m(e) \leq 4$ for all $e \in EG$. Moreover, m can be chosen so that if for some $e \in EG$ one holds $m(e) = 4$ then m satisfies the properties (i)-(iii) in Theorem 3.11.*

Fig. 3.7

Remark. A primitive triple $(G, \mathcal{H}, l)$ with $|\mathcal{H}| = 4$ and $l(e) = 4$ for some $e \in EG$ does exists, as shown in Fig. 3.7. Here $\mathcal{H} = \{I, J, K, O\}$ and the numbers on the edges indicate the values of l on them.

4. PROOF OF THEOREM 1. EXCLUSION OF $|B| = 2$

We use some ideas of the proof of Theorem 2 in [Ka2]. Without loss of generality one may assume that: G is connected; all $s_1, \ldots, s_r, t_1, \ldots, t_r$ are distinct and of valency 1 (since for each i one can add new vertices s'_i, t'_i and edges $\{s'_i, s_i\}$,$\{t'_i, t_i\}$ to G and consider the pairs $\{s'_i, t'_i\}$ instead of $\{s_i, t_i\}$). Let $T := \{s_1, \ldots, s_r, t_1, \ldots, t_r\}$. Also one may assume that each *inner* vertex x (i.e. $x \in VG - T$) is of valency 2 or 4 (otherwise one can repeatedly transform G at x as shown in Fig. 4.1; obviously, this does not change, in essense, our problem).

Fig. 4.1

Supposing that Theorem 1 is not true, we consider a counterexample with $|EG|$ as small as possible. Then G has neither loops nor inner vertices of valency 2.

For $x \in VG$ let $E(x)$ denote the set of edges of G incident to x and ordered clockwise in the plane. Consider $x \in VG - T$ and two consecutive edges $e, e' \in E(x)$. The triple $\tau = (e, x, e')$ is called a *fork*. Denote by G_τ the graph obtained from G by adding a new edge (or a loop) e_τ connecting the ends of the edges e and e' different from x. Define the function ω_τ on EG_τ by

$$\begin{aligned} \omega_\tau(u) &:= 1 \quad \text{for } u = e, e', \\ &:= -1 \quad \text{for } u = e_\tau, \\ &:= 0 \quad \text{otherwise.} \end{aligned}$$

For $0 \le \varepsilon \le 1$, let $c_{\tau,\varepsilon}$ denote the function on EG_τ taking the value $1 - \varepsilon$ on e and e', ε on e_τ, and 1 on the edges in $EG - \{e, e'\}$. We say that ε is *feasible* if the problem $(c_{\tau,\varepsilon}, d)$ has a solution. In particular, $\varepsilon = 0$ is feasible. The maximum feasible $\varepsilon \le 1$ is denoted by $\alpha(\tau)$.

Suppose that there is a fork $\tau = (e, x, e')$ with $\alpha(\tau) = 1$. Then one can split off e, e' at x preserving solvability of the problem. More precisely, let G' arise from G by deleting e,e' and adding e_τ. Since $|EG'| = |EG| - 1$ and $(G', U)^*$ is solvable, it has a half-integral solution; this easily implies that $(G, U)^*$ has a half-integral solution as well.

Thus, $\alpha(\tau) < 1$ for all forks τ in G. Consider a fork $\tau = (e, x, e')$; let y (z) be the end of e (e') different from x. For $\alpha(\tau) < \varepsilon \le 1$ the problem $(c_{\tau,\varepsilon}, d)$ has no solution, therefore, there is a metric m on VG for which

$$c_{\tau,\varepsilon}(m) - d(m) < 0. \tag{4.1}$$

Moreover, by Theorem 3 one can choose m which is either a cut metric or a 2,3-metric or a 4f-metric on $VG_\tau = VG$. Define $\omega_\tau(m) := m(e) + m(e') - m(e_\tau)$; then $\omega(m) \ge 0$

(since m is a metric). Clearly $c_{\tau,\varepsilon}(m) = c(m) - \omega_\tau(m)$, and now $c(m) \ge d(m)$ (as (c,d) is solvable) implies that $\omega_\tau(m) > 0$. Thus, the following is true.

Statement 4.1. *$\alpha(\tau)$ can be determined as*

$$\alpha(\tau) = \min\{(c(m) - d(m))/\omega_\tau(m)\}, \tag{4.2}$$

where the minimum is taken over all cut-, 2,3-, and 4f-metrics m for which $\omega_\tau(m) > 0$. •

A metric m that achives the minimum in (4.2) is called *critical* for τ.

Statement 4.2. *For any cut-, 2,3- or 4f-metric m the values $c(m) - d(m)$ and $\omega_\tau(m)$ are even.*

Proof. Since such a metric is induced by a bipartite graph, and $\omega_\tau(m) \equiv \sum(m(u) : u \in C) \pmod 2$, where C is the circuit formed by the edges e, e', e_τ, the value $\omega_\tau(m)$ is even. Next, the graph $(VG, EG \cup U)$ is eulerian, therefore it is represented as the union of pairwise edge-disjoint circuits $C_1, \ldots, C_k$. For each i the value $\sum(m(u) : u \in C_i)$ is even, and $c(m) - d(m) \equiv \sum_{i=1}^{k} \sum(m(u) : u \in C_i) \pmod 2$. Hence, $c(m) - d(m)$ is even. •

Notice that for any $u \in EG$ one has: $m(u) \le 1$ if m is a cut metric; $m(u) \le 2$ if m is a 2,3-metric; and, by Theorem 4, $m(u) \le 4$ if m is a 4f-metric. Hence,

$$\begin{aligned} \omega_\tau(m) &\in \{0,2\} && \text{if } m \text{ is a cut metric;} \\ &\in \{0,2,4\} && \text{if } m \text{ is a 2,3-metric;} \\ &\in \{0,2,4,6,8\} && \text{if } m \text{ is a 4f-metric.} \end{aligned} \tag{4.3}$$

Summing up (4.3) and Statements 4.1 and 4.2, we get the following.

Statement 4.3. *Let $0 < \alpha(\tau) < 1$, and let m be a metric critical for τ. Then:*

(i) m is not a cut metric;

(ii) if m is a 2,3-metric then $\alpha(\tau) = 1/2$ (cf. [Ka2]*);*

(iii) if m is a 4f-metric then $\alpha(\tau) \in \{\frac{1}{4}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}\}$, and in case $\alpha(\tau) = 3/4$ the equalities $m(e) = m(e') = 4$ and $m(y,z) = 0$ hold. •

The case $\alpha(\tau) = 3/4$ will be of most interest for us in what follows.

Now we continue considerations begun in Section 2. Let us fix f and B as in (2.5), and let h_F, $F \in B$, be functions satisfying the properties as in Lemma 2.2 and Statement 2.3. In view of Statements 2.1 and 2.4, one may assume that

(4.4) for any $F \in B$ the circuit C_F has at least one edge common with $C_{F'}$ for some $F' \in B - \{F\}$, and h_F is not equal identically to $1/2$ on C_F.

Following [Ka2], introduce the value $\beta(\tau)$ which, as we shall see later, gives a lower bound for $\alpha(\tau)$:

$$\beta(\tau) := 1 + f^{e,e'} - \frac{1}{2}f^e - \frac{1}{2}f^{e'} \quad \left(= 1 - \frac{1}{2}(f^{e,u} + f^{e,u'} + f^{e',u} + f^{e',u'})\right),$$

where $E(x) = (e, e', u, u')$, and for a pair e'', u'' of edges, $f^{e'',u''}$ denotes $\sum(f(L) : L \in \mathcal{L}, e'', u'' \in L)$. By symmetry,

$$\beta(e, x, e') = \beta(u, x, u'). \tag{4.5}$$

Statement 4.4 [Ka2]**.** $\beta(\tau) \le \alpha(\tau)$.

Proof. Let for definiteness $f^e \ge f^{e'}$. Define the capacity function c' on EG_τ as: $c'(e) := f^e - f^{e,e'}$; $c'(e') := f^{e'} - f^{e,e'}$; $c'(e_\tau) := 1 + f^{e,e'} - f^e$; and $c'(w) := c(w)$ for the other edges w. It is easy to see that (c', d) has a solution. Now put $c'' := c_{\tau,\beta(\tau)}$ and $\varepsilon := (f^e - f^{e'})/2$. A straightforward check-up shows that $c''(w) - c'(w)$ is equal to ε for $w = e', e_\tau$; $-\varepsilon$ for $w = e$; and 0 for the other $w \in EG_\tau$. Since $\varepsilon > 0$, the solvability for (c', d) implies that for (c'', d). Hence $\alpha(\tau) \ge \beta(\tau)$. •

Remark 4.5. Statements 4.3 and 4.4 imply that if $\beta(\tau) = 3/4$ holds for a fork τ then $\alpha(\tau) = \beta(\tau)$. Moreover, from the proof of Statement 4.4 one can see that in this case f can be easily transformed locally, "within the edges e, e', e_τ", to give a solution f' for $(c_{\tau,3/4}, d)$. More precisely, let $f^e \ge f^{e'}$ and $P \in \mathcal{L}$. If $e \notin P$, put $f'(P) := f(P)$. If $e, e' \in P$ then P is transformed into P' with $f'(P') := f(P)$ by replacing e, e' by the edge e_τ. If $e \in P \not\ni e'$ then put $f'(P) := f(P)\left(\frac{1}{2}f^e + \frac{1}{2}f^{e'} - f^{e,e'}\right)/(f^e - f^{e,e'})$ and form a new path P' from P by replacing e by the edges e', e_τ, for which one puts $f'(P') := f(P)\left(\frac{1}{2}f^e - \frac{1}{2}f^{e'}\right)/(f^e - f^{e,e'})$. One can check that f' is $(c_{\tau,3/4}, d)$-admissible and gives a solution for $(c_{\tau,3/4}, d)$. Note also that if m is critical for τ then m is a 4f-metric, by Statement 4.3. Moreover, $c_{\tau,3/4}(m) = d(m)$ and the fact that f' is a solution for $(c_{\tau,3/4}, d)$ imply that

(4.6) each edge $w \in EG_\tau$ with $m(w) > 0$ is saturated by f' (i.e. $(f')^w = c_{\tau,3/4}(w)$) and every path $P \in \mathcal{L}(f')$ is shortest for m.

Return to consideration of the bunch B as above.

Statement 4.6.

(i) If e is a common edge for C_F and $C_{F'}$ $(F, F' \in B)$ and $h_F(e) = h_{F'}(e) = 0$ then $(G, U)^*$ has a half-integral solution.

(ii) If e, e' are two edges incident to a vertex x, and if $e \in C_F$ and $e' \in C_{F'}$ for distinct $F, F' \in B$ and $h_F(e) = h_{F'}(e) = 0$, then $(G, U)^*$ has a half-integral solution.-

Proof. (i) By (2.2), e is a common edge in the boundaries of F and F'. Delete e from G, forming G'; as a result, the holes F and F' merge into one face. Clearly f gives a solution for $(G',U)^*$. Since the number of holes for G',U is three then, by Theorem 2, $(G',U)^*$ has a half-integral solution (not necessarily an integral one because the graph $(VG',EG'\cup U)$ is not eulerian). This implies (i).

(ii) By (2.2), $e\in \mathrm{bd}(F)$ and $e'\in \mathrm{bd}(F')$. Obviously, G can be splitted at x in such a way that the holes F and F' merge into one face of the resulting graph G', and f gives a solution for $(G',U)^*$. Now we apply arguments as in the proof of (i). •

Thus, the situation as in the hypotheses of Statement 4.4 cannot occur. The following statement of topologic character will play important role as being a tool that will be often used in what follows. Its proof appeals to (3.6),(4.6) and simple topological observations, and we leave it to the reader.

Statement 4.7. *Let f' be a solution for some G',c',d', and let B be a bunch for f' (assuming that f' is non-crossing). Let m' be a 4f-metric on VG' induced by $\sigma: VG'\to VH$ such that $c'(m')=d'(m')$. Next, let $C_F=(v_0,e_1,v_1,\ldots,e_k,v_k)$ be a circuit in $\mathcal{C}(B)$, and let C be its image (in the sense of (3.5)) in H. Then C is a simple circuit, and C_F separates holes $F',F''\in\mathcal{H}$ if and only if C separates the corresponding faces $\sigma(F'),\sigma(F'')$ in H.* •

Lemmas 4.8 and 4.9 bellow will be key in this section. Obviously, one may assume that

$$h_F(e)=\frac{1}{2}\lceil 2\overline{f}^e_F\rceil \ \text{ for any } \ e\in C_F, F\in B \tag{4.7}$$

($\overline{f}^e_F$ was defined before Lemma 2.2). For a vertex x in C_F ($F\in B$) let $E_F(x)$ denote the set of edges incident to x and contained in Ψ_F-C_F (then $|E_F(x)|\le 2$).

Lemma 4.8 . *Let L be a maximal path in $C_F\cap C_{F'}$ ($F,F'\in B$). Then either $h_F(e)=h_{F'}(e)=1/2$ for all $e\in L$, or $h_F(e)=0$ and $h_{F'}(e)>0$ for all $e\in L$, or $h_{F'}(e)=0$ and $h_F(e)>0$ for all $e\in L$.*

Proof. Let each of h_F and $h_{F'}$ be not equal to zero identically on L. One must prove that $h_F(e)=h_{F'}(e)=1/2$ for all $e\in L$. Suppose this is not so. Then for some $F''\in\{F,F'\}$ there is a pair e,e' of consecutive edges in L such that $h_{F''}(e)\ne 0=h_{F''}(e')$; let for definiteness $F''=F$. By Statement 4.6, $h_{F'}(e)\ne 0\ne h_{F'}(e')$, hence $h_F(e)=h_{F'}(e)=1/2$ and $h_{F'}(e')\in\{\frac{1}{2},1\}$. Let x be a vertex incident to e and e'. Since $h_F(e)\ne h_F(e')$, $E_F(x)\ne\emptyset$ (in view of (4.7)). Consider two possible cases.

Case 1. $|E_F(x)|=1$. Let for definiteness $F=I$, $F'=J$, $E(x)=(e,u,e',u')$ and $E_I(x)=\{u\}$; see Fig. 4.2. Clearly $f^{u,e'}=f^{u,u'}=0$. Also $f^{e,e'}+f^{e,u'}\le 1/2$ (as any path in $\mathcal{L}(f)$ passing through e and some of e',u' concerns the flow $\overline{f}_J$, and the total amount of flow over these paths is at most $h_J(e)=1/2$). Hence, for the fork $\tau=(e,x,u)$ one has $\alpha(\tau)=\beta(\tau)=3/4$.

Consider the solution f' for $G_\tau, c_{\tau,3/4}, U$ obtained from f as in Remark 4.5 (for $\tau = (e,x,u)$), and a 4f-metric m critical for τ. By Statement 4.3(iii), $m(e) = m(u) = 4$ and $m(y,z) = 0$, where y (z) is the end of e (u) different of x. One may assume that m is induced by a mapping $\sigma : VG_\tau \to VH$ as in Theorem 3.11, and $\sigma(x) = b_4$, $\sigma(y) = \sigma(z) = b_0$. We observe that there is a path in $\mathcal{L}_J(f')$ passing through x and y. Hence, by (4.6) and (i),(iii)(a) in Theorem 3.11, $\sigma(J) = \widetilde{J}$. On the other hand, $e' \in \mathrm{bd}(I)$ (as $f_I^{e'} \le h_I(e') = 0$), whence $u = xz \in \mathrm{bd}(I)$ (as u lies in Ψ_I and u, e' are consecutive in $E(x)$). This implies that $b_0 = \sigma(z)$ and $b_4 = \sigma(x)$ belong to the boundary of $\sigma(I)$ in H. Hence, b_0, b_4 belong to a shortest $\sigma(I)$-path; a contradiction with (iii)(a) in Theorem 3.11.

Fig. 4.2

Case 2. $|E_F(x)| = 2$. Let for definiteness $F = J$, $F' = I$, $E(x) = (e, u, u', e')$; then $E_J(x) = \{u, u'\}$; see Fig. 4.3a. Since $E_I(x)$ is empty, $h_I(e) = h_I(e') = 1/2$ (in view of (4.7)). We observe that $\beta(\tau) = 3/4$ for $\tau = (e, x, e')$ (since $f^{e',u} = f^{e',u'} = 0$ and $f^{e,u} + f^{e,u'} \le h_J(e) = 1/2$).

Fig. 4.3

Consider the solution f' for $G_\tau, c_{\tau,3/4}, U$ as in Remark 4.5, and a 4f-metric m critical for τ. By Statement 4.3(iii), $m(e) = m(e') = 4$ and $m(y,z) = 0$, where y

(z) is the end of e (e') different of x. One may assume that m is induced by $\sigma : VG_\tau \to VH$ as in Theorem 3.11, and that $\sigma(x) = b_4$, $\sigma(y) = \sigma(z) = b_0$. One can see that $(f')^e_J = (f')^{e'}_J > 0$ (taking into account that $h_J(e) = 1/2$, $h_J(e') = 0$ and $c_{\tau,3/4}(e) = c_{\tau,3/4}(e') = 1/4$), whence $\sigma(J)$ coincides with $\widetilde{J}$ as in Theorem 3.11. Next, since $(f')^{e_\tau}_I > 0$ (as $h_I(e) = h_I(e) = 1/2$), b_0 belongs to a shortest $\sigma(I)$-path in H. However, in contrast to the previous case, we cannot assert that both b_0, b_4 belong to a shortest $\sigma(I)$-path. To this reason, we need additional arguments that rely on Statement 4.7.

One may assume that $m(w) \le 1$ for each $w \in EG_\tau - \{e, e'\}$ (for if $m(w) = k > 1$ for some w then one can subdivide w into k edges in series, each of m-length 1, accordingly defining f' and σ for the resulting graph).

Form the circuit C'_J in G_τ corresponding to C_J in G; clearly C'_J is obtained from C_J be replacing e, e' by e_τ. Put $X := \sigma^{-1}(b_4)$ and $Y := \sigma^{-1}(b_0)$. Apply Statement 4.7 to $G_\tau, c_{\tau,3/4}, d, f'$ and C'_J. It follows from (ii) in Theorem 3.11 that the image C in H of C'_J is the circuit forming the boundary of $\widetilde{J}$. Therefore, by Statement 4.7, the closed region Ψ'_J bounded by C'_J and containing J does not contain any other hole. Let $\overline{G}$ be the subgraph of G_τ lying in Ψ'_J.

By the property (3.4) of a 4f-metric, one can span X by a tree T_X whose edges are embedded (with preserving planarity) in intermediate faces of G_τ. As T_X connects J with another hole, X meets C'_J in some vertex x'. Moreover, some vertex $v \in \mathrm{bd}(J) \cap X$ is connected with x by a path L and connected with x' by a path L' such that L, L' lie in $T_X \cap \overline{G}$.

Consider the region $\Omega \subset \Psi'_J$ that contains no hole and is bounded by L, L' and the part D of C_J from x to x'; see Fig 4.3b. Next, consider two consecutive vertices p, q in $D - \{x\}$ such that $p \in Y \not\ni q$. Then $m(p, q) = 1$; let $b := \sigma(q)$. We observe that $b \in \{b_0, \dots, b_4\}$. Indeed, the set $Q := \sigma^{-1}(b) \cap V\overline{G}$ can be spanned by a tree T of new edges lying in Ψ'_J and intersecting neither edges of G_τ nor vertices in $X \cup Y$. Hence, $T \subset \Omega - \{L, L'\}$. But $b \notin \{b_0, \dots, b_4\}$ would imply that Q contains a vertex in $\mathrm{bd}(J)$; a contradiction.

Thus, $b = b_1$ (since $\sigma(p) = b_0$ and $m(p, q) = 1$). Obviously, the edge $w = pq$ cannot belong to $\mathrm{bd}(J)$, whence $(f')^w_J > 0$. Choose a J-path $P = z_0 \dots z_r$ in G_τ such that $f'(P) > 0$, $z_i = q$, $z_{i+1} = p$, and let P' be its part from q to z_r. Since b_1, b_0, b_4 cannot occur (in this order) in any shortest path in H, P' must pass through the edge e_τ. On the other hand, from the construction of f' (see Remark (4.5)) and the facts that $h_J(e) = 1/2$ and $h_J(e') = 0$ it follows that any path in $\mathcal{L}_J(f')$ containing e_τ traverses e or e'. This again implies that P is not shortest for m; a contradiction. •

Lemma 4.9. *Let $F, F' \in B$, and let $P = (v_0, e_1, v_1, \dots, e_k, v_k)$ and $P' = (v'_0, e'_1, v'_1, \dots, e'_q, v'_q)$ be paths (possibly circuits) in C_F and $C_{F'}$, respectively, such that $v_0 = v'_0$, $e_1 = e'_1$, $e_2 \ne e'_2$ and $v_k = v'_q$. Let $h_F(e_1) = h_{F'}(e_1) = 1/2$. Let the region bounded by P and P' (outside Ψ_F and $\Psi_{F'}$) contain no hole. Then $(G, U)^*$ has a half-integral* *solution.*

Proof. Put $e := e_1$, $x := v_1$, $e' := e_2$, $u' := e_2'$. Since e, e', u' are distinct, $|E_F(x)| + |E_{F'}(x)| \le 1$. Therefore, one may assume that $E_F(x) = \emptyset$. Let for definiteness $F = I$ and $F' = J$. We observe that $h_I(e') = h_I(e) = 1/2$, that $f^{e',u} + f^{e',u'} = 0$ (taking into account that $f^{e'} = \overline{f}_I^{e'}$ and $f^{u'} = \overline{f}_J^{u'}$ since there is no hole between P and P'), and that $f^{e,u} + f^{e,u'} \le h_J(e) = 1/2$, where $\{u\} := E(x) - \{e, e', u'\}$. Hence, $\beta(\tau) = 3/4$ for $\tau = (e, x, e')$. Let f' be the solution for $G_\tau, c_{\tau,3/4}, U$ as in Remark 4.5, and let m be a 4f-metric critical for τ and induced by $\sigma : VG_\tau \to VH$ as in Theorem 3.11. Let for definiteness $\sigma(x) = b_4$ and $\sigma(y) = \sigma(z) = b_0$, where y (z) is the end of e (e') different of x, see Fig. 4.4.

Fig. 4.4

One can see that $(f')^e_J, (f')^{e'}_J > 0$ and that the corresponding circuit C'_I for f' is formed from C_I by replacing e, e' by e_τ, while C'_J is formed from C_J by replacing e by e', e_τ. Hence, by (i),(iii)(a) in Theorem 3.11, $\sigma(J) = \widetilde{J}$. By (ii) in Theorem 3.11, the image C of C'_J is the circuit in H forming the boundary of $\widetilde{J}$. In addition, b_0 belongs to a shortest $\widetilde{I}$-path, where $\widetilde{I} := \sigma(I)$ (since $y \in C'_I$).

Hence, b_4 belongs to a shortest $\widetilde{I}$-path. By (iii)(a) in Theorem 3.11, some $b \in \{b_0, b_4\}$ is not in $\mathrm{bd}(\widetilde{I})$. Let $w \in \{x', y\}$ be the vertex such that $\sigma(w) = b$. Since $\sigma^{-1}(b)$ does not meet $\mathrm{bd}(I)$, and w is in C'_I, there is an I-path Q in G_τ passing through w and such that $f'(Q) > 0$. Hence, the image L of Q is a shortest $\widetilde{I}$-path in H. Moreover, L separates $\widetilde{J}$ from $\widetilde{K}$ and $\widetilde{O}$ (by (iii)(b) in Theorem 3.11), whence $\mathcal{F}_H = \{\widetilde{I}, \widetilde{J}, \widetilde{K}, \widetilde{O}\}$. In view of Statement 4.7, this means that C'_I (and hence C_I too) separates J from K and O, whence we conclude that $|B| = 2$, $|\mathcal{H}_I| = 3$ and $|\mathcal{H}_J| = 1$. On the other hand, (iii)(c) in Theorem 3.11 implies that C_{IK} does not separate K and O. Hence, there is a bunch B' for f such that either $|B'| = 3$, or $|B'| = 2$ and $\{|\mathcal{H}_F| : F \in B'\} = \{2, 2\}$. A contradiction with the choice of B in (2.5).

Statement 4.10. *Let $F \in B$ and let $\{P_1, \dots, P_k\}$ be the set of all maximal nontrivial paths in C_F such that for $i = 1, \dots, k$, P_i is a part of $C_{F(i)}$ for some $F(i) \in B - \{F\}$. Let either $h_F(e) = 0$ for all $e \in P_i$ or $h_{F(i)}(e) = 0$ for all $e \in P_i$, $i = 1, \dots, k$. Then*

$(G,U)^*$ *has a half-integral solution.*

Proof. Let N be the set of $i \in \{1,\dots,k\}$ for which $h_F(e)=0$ for all $e \in P_i$. Define capacities c' on EG_F by

$$\begin{aligned} c'(e) &:= 0 \quad \text{if } e \in P_i \text{ and } i \in N, \\ &:= 1 \quad \text{otherwise,} \end{aligned}$$

and define capacities c'' on $EG-(EG_F-C_F)$ by

$$\begin{aligned} c''(e) &:= 0 \quad \text{if } e \in P_i \text{ and } i \notin N, \\ &:= 1 \quad \text{otherwise.} \end{aligned}$$

Then $c'(e)+c''(e) \le 1$ for each $e \in C_F$. Since c' is integral, and $|\mathcal{H}_F| \le 3$, the problem for c' and U_F has a half-integral solution, and similarly for c'' and $U-U_F$, whence the result follows. •

Now Lemmas 4.8, 4.9 and Statement 4.10 enable us to exclude case $|B|=2$, as follows. Let for definiteness $B=\{I,J\}$, and let $\{P_1,\dots,P_k\}$ be the set of maximal nontrivial paths in $C_I \cap C_J$. The result follows immediately from Lemmas 4.8 and 4.9 if for some i one has $c_I(e)=c_J(e)=1/2$ for $e \in P_i$. Otherwise, by Lemma 4.8, there is $N \subseteq \{1,\dots,k\}$ such that: if $i \in N$ then $c_I(e)=0$ for all $e \in P_i$, and if $i \notin N$ then $c_J(e)=0$ for all $e \in P_i$. In this case the result follows from the Statement 4.10.

5. EXCLUSION OF $|B|=4$

We assume that $|B| \ge 3$. First of all we state several statements which will be used for both cases $|B|=4$ and $|B|=3$ (the latter is considered in the next section).

Lemma 5.1. *Let $F,F',F'' \in B$, and let e,u be two consecutive edge in C_F such that $e \in C_{F'}$ and $u \in C_{F''}$. Let x be a common vertex for e,u, and let e' (u') be the edge in $C_{F'}$ ($C_{F''}$) incident to x and different of e (u). Then:*

(i) $h_F(e)=h_{F'}(e)=h_F(u)=h_{F''}(u)=1/2$;

(ii) if either $|B|=3$, or $|B|=4$ and x does not belong to $C_{\widetilde{F}}$, where $\{\widetilde{F}\}=B-\{F,F',F''\}$, then $e'=u'$ (and therefore, $h_{F'}(e')=h_{F''}(e')=1/2$).

Proof. Suppose that (i) is false. Let for definiteness $F=I$, $F'=J$, $F''=K$. By Statement 4.6, among $h_I(e),h_I(u),h_J(e),h_J(e'),h_K(u),h_K(u')$ there are no zero numbers $h_Q(q),h_{Q'}(q')$ for $Q \ne Q'$. In particular, it is impossible that $h_I(e)=h_I(u)=1$ (as then $h_J(e)=h_K(u)=0$), or $h_I(e)=0$ and $h_I(u)=1$ (as then $h_K(u)=0$). Consider possible cases (we omit symmetric cases).

(a) $h_I(e)=h_I(u)=0$. If $e' \ne u'$ then $\beta(\tau)=1$ for $\tau=(e,x,e')$ (as, obviously, $f^{e,u}=f^{e,u'}=f^{e',u}=f^{e',u'}=0$). Hence, $e'=u'$. Let for definiteness $E_J(x)=\emptyset$.

Since $h_J(e') \neq 0$ and $h_K(e') \neq 0$, we have $h_J(e') = h_K(e') = 1/2$. This implies that $h_J(e) = 1/2$ and $\beta(\tau) = 3/4$ for $\tau = (e, x, e')$. Let f' be the solution for $G_\tau, c_{\tau,3/4}, U$ as in Remark 4.5, and let m be a 4f-metric critical for τ and induced by $\sigma : VG_\tau \to VH$ as in Theorem 3.11. Then $\{\sigma(x), \sigma(y) = \sigma(z)\} = \{b_0, b_4\}$, where y (z) be the end of e (e') different of x. Since $x, y \in \mathrm{bd}(I)$, $\sigma(I) = \widetilde{J}$ (by (i),(iii)(a) in Theorem 3.11). On the other hand, from construction of f' and the facts that $1 = f^{e'} > f^e = 1/2$ and $f_K^{e'} > 0$ it follows that there exists a K-path P in G_τ containing e and such that $f'(P) > 0$. Hence, the image of P is a shortest $\sigma(K)$-path in H containing b_0, b_4; a contradiction with (iii)(a) in Theorem 3.11.

(b) $h_I(e) = 1/2$ and $h_I(u) = 0$. Then $|E_I(x)| = 1$; let $E_I(x) = \{e''\}$. It is easy to see that $\beta(\tau) = 3/4$ for $\tau = (e, x, e'')$. A contradiction is shown similarly to as it is done in Case 1 in the proof of Lemma 4.8.

(c) $h_I(e) = 1$ and $h_I(u) = 1/2$. Then $|E_I(x)| = 1$, whence $e' = u'$. Moreover, from the facts that $h_I(e) = 1$ and $E_J(x) = \emptyset$ it follows that $h_J(e) = h_J(e') = 0$. Hence, this case is similar to (a).

(d) $h_I(e) = h_I(u) = 1/2$ and $h_J(e) = 0$. If $E_I(x) \neq \emptyset$ then $E_J(x) = \emptyset$, $e' = u'$ and $h_J(e') = 0$; so this case is similar to (a). Let $E_I(x) = \emptyset$. From $h_J(e) = 0$ it follows that $\beta(\tau) = 3/4$ for $\tau = (e, x, u)$; further arguments are similar to those applied in case (a) (for I instead of J).

To prove (ii), suppose, for converse, that $e' \neq u'$, and consider the fork $\tau = (e, x, e')$. From (i) and the hypothesis in (ii) it easily follows that $\beta(\tau) = 3/4$. Now a contradiction is shown similarly to as it is done in Case 1 in the proof of Lemma 4.8. •

The following statement strengthens Statement 2.3 in case $|\mathcal{H}_F| = 1$.

Statement 5.2. *Let $F \in B$ and $|\mathcal{H}_F| = 1$. Then each edge $e \in C_F$ belongs to a tight cut for G_F, h_F, d_F.*

Proof. In view of Statement 2.3, it suffices to consider $e \in C_F$ with $h_F(e) = 0$. Then $e \in \mathrm{bd}(F)$. Suppose that the statement for e is wrong. Then

$$h_F(X) - d_F(X) \geq \frac{1}{2} \quad \text{for any } X \subseteq VG_F \tag{5.1}$$

($h'(X)$ and $d'(X)$ are defined as in the proof of Lemma 2.2)). Let x and y be the ends of e. Add to U_F one more demand pair $w = \{x, y\}$ for which we put demand $d_F(w) = 1/2$. In view of (5.1), from Okamura's theorem it follows that the problem (h_F, d') (where d' denotes the demand function on $U_F \cup \{w\}$) has a solution f'. Let L be a path with $f'(L) > 0$ connecting x and y. Taking into account that $e \in \mathrm{bd}(F) \cap C_F$, one can see that every cut δX which meets both $\mathrm{bd}(F)$ and C_F must have a common edge with L, therefore $h_F(X) > d_F(X)$. This implies that no edge in C_F belongs to a tight cut for h_F and d_F, whence, by Statement 2.3, $h_F(e') = 0$ for all $e' \in C_F$. Then, obviously, $(G, U)^*$ has a half-integral solution. •

We call a 1/2-*segment* for $F \in B$ a maximal non-trivial path P in C_F with $h_F(e) = 1/2$ for all $e \in P$; let ω_F denote the number of 1/2-segments in C_F. A cut δX in G_F is called *simple* if the subgraphs of G_F induced by X and $VG_F - X$ are connected. If δX is simple then, obviously, $|\delta X \cap \mathrm{bd}(F')| \in \{0,2\}$ for every face F' in G_F. Clearly if $\delta X'$ is tight (for G_F, h_F, U_F) then $\delta X' = \delta X_1 \cup \ldots \cup \delta X_k$, where each δX_i is a simple tight cut. In what follows by a cut we shall always mean a simple cut.

Consider some F with $|\mathcal{H}_F| = 1$. The following lemma plays important role in the proof.

Lemma 5.3.

(i) *For G_F, h_F, U_F each tight cut meets each 1/2-segment in C_F at most in one edge.*

(ii) *ω_F is even.*

(iii) *If $S_0, S_1, \ldots, S_{2k-1}$ are the 1/2-segments occurring in this order in C_F then every tight cut meeting some S_i meets S_{i+k} (indices are taken modulo $2k$).*

Proof. Let for definiteness $F = I$. Consider a tight cut δX with $|\delta X \cap C_I| = 2$; let $\{e, e'\} = \delta X \cap C_I$. This cut can be naturally associated with the dual curcuit (or the circuit of the dual graph) $D_X = (F_0, e_1, F_1, \ldots, e_k, F_k)$ for which $F_0 = F_k = \widetilde{F}$, $e_1 = e$ and $e_k = e'$, where $\widetilde{F}$ is the face in G_F bounded by C_F. Let $F_i = I$. We define a partition D_X into two dual paths:

$$D_X(e) := (F_0, e_1, \ldots, F_i) \quad \text{and} \quad D_X(e') := (F_i, e_{i+1}, \ldots, F_k).$$

Since $d_I(X)$ is an integer and $h_I(e_j)$ is an integer for $j = 2, \ldots, k-1$, we have

(5.2) either $h_I(e), h_I(e') \in \frac{1}{2}$ or $h_I(e), h_I(e') \in \{0,1\}$.

First of all we prove two claims.

Claim 1. *Let $\delta X, \delta Y$ be two tight cuts such that $\delta X \cap C_I = \{u, u'\}$, $\delta Y \cap C_I = \{z, z'\}$, $h_I(u), h_I(u') \in \frac{1}{2}$ and $h_I(z), h_I(z') \in \{0,1\}$. Then D_X and D_Y have no common faces except I and $\widetilde{F}$.*

Proof. Consider the dual paths $D_X(u), D_X(u')$ in $D_X = (F_0, e_1, F_1, \ldots, e_k, F_k)$ and the dual paths $D_Y(z), D_Y(z')$ in $D_Y = (F'_0, e'_1, F'_1, \ldots, e'_{k'}, F'_{k'})$. Suppose, e.g., that $D_X(u) = (F_0, e_1, \ldots, e_j, F_j)$ and $D(z) = (F'_0, e'_1, \ldots, e_{j'} F'_{j'})$ have a common face $F_i = F'_{i'}$ different from I and $\widetilde{F}$. Put $E_1 := \{e_1, \ldots, e_j, e'_{j'+1}, \ldots, e'_{k'}\}$ and $E_2 := \{e'_1, \ldots, e'_{j'}, e_{j+1}, \ldots, e_k\}$. It is not difficult to show that there are tight cuts $\delta X' \subseteq E_1$ and $\delta Y' \subseteq E_2$ such that $\delta X'$ contains $e_1 = u$ and $e'_{k'} = z'$, while $\delta Y'$ contains $e'_1 = z$ and $e_k = u'$. Since $h_I(u) = 1/2$ and $h_I(z') \in \{0,1\}$, we get a contradiction with (5.2) (for $\delta X'$, $e := u$, $e' := z'$). •

Claim 2. *Let $\delta X, \delta Y, u, u', z, z'$ be as in the hypotheses in Claim 1. Then the pairs*

$\{u,u'\}$ *and* $\{z,z'\}$ *are crossing in* C_I *(that is, up to permutation of* u,u' *and permutation of* z,z'*, these edges occur in* C_I *in order* u,z,u',z'*).*

Proof. Suppose, e.g., that these edges occur in C_I in order u,u',z,z' (clockwise from a point a in $\widetilde{F}$). Let $\overline{u},\overline{u}',\overline{z},\overline{z}'$ be the edges in $D_X(u),D_X(u'),D_Y(z),D_Y(z')$, respectively, that belong to $\mathrm{bd}(I)$. From Claim 1 it follows that the latter edges occur in $\mathrm{bd}(I)$ in order $\overline{u},\overline{u}',\overline{z},\overline{z}'$ (clockwise from a). Let $\delta X'$ ($\delta Y'$) be the cut formed by the edges in $D_X(u)\cup D_Y(z)$ (resp., $D_X(u')\cup D_Y(z')$). One can see that $d_I(X')+d_I(Y')\geq d_I(X)+d_I(Y)$, whence we conclude that $\delta X'$ and $\delta Y'$ are tight. A contradiction with (5.2). •

Now suppose that there are a tight cut δX and a 1/2-segment S having two common edges u,u'. Since $S\neq C_I$ (by Statement 2.4), there is an edge $z\in C_I$ with $h_I(z)\in\{0,1\}$. By Statement 5.2, z belongs to a tight cut δY; let $\delta Y\cap C_I=\{z,z'\}$. By (5.2), $h_I(z')$ is an integer, so $z'\notin S$. This contradicts to Claim 2 and proves (i).

Let us prove (ii)-(iii). From (i) and (5.2) it follows that $\omega_I\geq 2$ and that (iii) is true for $\omega_I=2$. Let $C_I=S_0\cdot L_0\cdot S_1\cdot L_1\cdot\ldots\cdot S_{k'}\cdot L_{k'}$ ($k'=\omega_I-1\geq 2$), where each S_i is a 1/2-segment. It is easy to see from (i) that if (ii) or (iii) is not true then there are indices (up to a cyclical shift) $0\leq i\leq i'<j'<j\leq k'$ and tight cuts $\delta X,\delta X'$ so that δX meets S_i and S_j while $\delta X'$ meets $S_{i'}$ and $S_{j'}$. Choose an edge $z\in L_{j'}$ and a tight cut δY containing z. Clearly at least one of the pairs $\{\delta X,\delta Y\}$ and $\{\delta X',\delta Y\}$ contradicts to Claim 2. ••

An edge $e\in C_F$ is called a 1-*edge* if $e\notin C_{F'}$ for any $F'\in B-\{F\}$, and a 2-*edge* otherwise. A maximal non-trivial path in C_F all edges of which are 1-edges (resp., 2-edges common for C_F and $C_{F'}$ for some (fixed) $F'\in B-\{F\}$) is called a 1-*path* (resp., a 2-*path*). A 2-path P is called *strong* if for some (or, in view of Lemma 4.8, for any) edge $e\in P$ one has $h_F(e)=h_{F'}(e)=1/2$, where $F'\neq F$ and $e\in C_{F'}$; and P is called *weak* otherwise. Note that each strong path P is contained in some 1/2-segment S (but, in general, P and S may not coincide).

We say that a strong path in S_F is *reducible* if it belongs to a 1/2-segment S such that the 1/2-segment S' opposite (in the sense of Lemma 5.3) to S consists only of 1-edges. Define the function $\overline{h}_F$ on EG_F by

$$
(5.3)\qquad
\begin{aligned}
\overline{h}_F(e) &:= 0 && \text{if } e \text{ belongs to a reducible (strong) path in } C_F,\\
&:= 0 && \text{if } e \text{ belongs to a weak path and } h_F(e)=0,\\
&:= \frac{1}{2} && \text{if } e \text{ belongs to a non-reducible (strong) path in } C_F,\\
&:= 1 && \text{otherwise.}
\end{aligned}
$$

One can see that

(5.4) for each $F\in B$ the problem $(\overline{h}_F,d_F)$ is solvable.

This implies that

(5.5) if for some $F \in B$ every strong path in C_F is reducible then $(G,U)^*$ has a half-integral solution.

Thus, each C_F contains a non-reducible (strong) path. Moreover,

(5.6) there are at least two different 1/2-segments in C_F containing non-reducible paths.

In order to exclude $|B| = 4$ (and also $|B| = 3$ later) we need one more statement. We say that two elements $x, y \in VG \cup EG$ are *dually connected* if they belong to the boundary of the same intermediate face in G.

Lemma 5.4. *Let $P = x_1 \dots x_k$, $P' = y_1 \dots y_r$, $P'' = z_1 \dots z_q$ be 1-paths in $C_F, C_{F'}$, $C_{F''}$ for distinct $F, F', F'' \in B$ so that $x_1 = y_r$, $y_1 = z_q$, $z_1 = x_k$. Let C_F and $C_{F'}$ have a common edge e with an end at x_1 for which $h_F(e) = h_{F'}(e) = 1/2$. Then there exists $1 < i \le q$ such that for the edge $u = z_{i-1}z_i$ one holds:*

(i) $h_{F''}(u) = 1$;

(ii) u is dually connected with x_1.

Proof. Let $F = I$ and $F' = J$. One may assume that e and $e' = xx_2$ are consecutive edges in $E(x)$, where $x := x_1$. Then $h_I(e') = 1/2$ and $\beta(\tau) = 3/4$ for $\tau = (e, x, e')$. Let f' be a solution as in Remark 4.5 for $G_\tau, c_{\tau,3/4}$. Let C'_J be the circuit for G_τ, f' corresponding to C_J; then C'_J is formed from C_J by replacing e by e', e_τ, see Fig. 5.1.

Fig. 5.1

Next, let m be a critical 4f-metric for τ induced by $\sigma : VG_\tau \to VH$ as in Theorem 3.11. Then $f_J^{e'} > 0$, whence, by (i),(iii)(a) in Theorem 3.1, $\sigma(J) = \widetilde{J}$. Furthermore, the region $\Omega \subset \mathbb{R}^2$ bounded by P, P', P'' contains no hole, therefore, as it follows from Statement 4.7, the closed path that is the image of the circuit $P \cdot P'' \cdot P'$ does not separate any two face of H. This implies that there is a vertex $x' \in \{x_3, \dots, x_k, z_2, \dots, z_{q-1}\}$ such that $\sigma(x') = \sigma(x)$. Notice that $x' = x_j$ (for some j) is impossible; otherwise we get a contradiction using arguments as in the proof of Lemma 4.9.

Hence, $x' = z_i$ for some $1 < i < q$. Let i be chosen minimum subject to $\sigma(z_i) = b_0$

(letting for definiteness that $\sigma(x) = b_0$). Then $\sigma(z_{i-1}) \in \{b_1, b_3\}$ (assuming, without loss of generality, that $m(w) \le 1$ for all $w \in EG_\tau - \{e, e'\}$. Now the result follows from the facts that each edge $w \in EG_\tau$ with $m(w) > 0$ must be saturated by f', that for any edge w' in the interion of Ω we have $(f')^{w'} = 0$, and that the image of each of $\widetilde{P}, P', P''$ is a simple path in H, where $\widetilde{P}$ is the part of P from x_2 to x_k (the latter follows from Statement 4.7). •

Now we begin to consider the case $|B| = 4$. Denote by Q the graph that is the union the circuits C_F, $F \in B$, and denote by Q' the graph that is obtained from Q by shrinking each 1-edge; let μ be the natural mapping of Q to Q'. For $F \in B$ let $\mathcal{R}(F)$ denote the set of all maximal paths $P = v_0v_1 \dots v_k$ in C_F such that: (i) v_0v_1 and $v_{k-1}v_k$ are 2-edges, (ii) there is $F' \in B - \{F\}$ such that each 2- edge in P belongs to $C_{F'}$, (iii) for each 1-edge $e \in P$ (if any) there is a simple circuit C in $P \cup C_{F'}$ such that C contains e and one component in $\mathbb{R}^2 - C$ contains no hole. It is easy to see that $|\mathcal{R}(F)|$ is equal to the number of essential (i.e. of degree ≥ 3) vertices in $\mu(C_F)$.

First of all we observe that:

(5.7) for each $P \in \mathcal{R}(F)$ either P is a strong 2-path or every 2-path in P is weak (in view of Lemma 4.9);

(5.8) $\mathcal{R}(F)$ contains at least two non-reducible (strong) 2- paths so that the 1/2-segments in C_F containing these paths are distinct, and if there are exactly two non-reducible paths then their 1/2-segments are opposite (in view of (5.6)).

In particular, $|\mathcal{R}(F)| \ge 2$ for any $F \in B$, therefore, Q' is of one of types Q'_1, Q'_2, Q'_3, Q'_4, as drawn in Fig. 5.2.

Fig. 5.2

Let Z denote the set of essential vertices in Q', and Z^0 denote the set of $x \in Z$ for which $\mu^{-1}(x)$ consists of a unique vertex in Q. For $F \in B$ we keep notation F for the corresponding faces in Q and Q'.

In the sequal of the proof we use one sort of transformations of functions h_F, as follows. Let $\pi = (P_1, \dots, P_k)$ be a sequence of some paths in C_F and $\rho = (*_1, \dots, *_k)$

be a sequence of signs + or −. We say that a function h' on EG_F is *formed from* h_F *by use of* π *and* ρ if

$$
(5.9) \qquad h'(e) := 1 \quad \text{if } e \in P_i \text{ and } *_i = +, \\
:= 0 \quad \text{if } e \in P_i \text{ and } *_i = -, \\
:= \overline{h}_F(e) \quad \text{otherwise,}
$$

where $\overline{h}_F$ is defined as in (5.3).

Statement 5.5. *Suppose that for each* $F \in B$ *there are no two non-reducible (strong) paths which belong to the same 1/2-segment in* C_F. *Then* $(G,U)^*$ *has a half-integral solution.*

Proof. Consider a sequence $\xi = (L_0, F_1, L_1, \dots, F_r, L_r)$, $r \geq 1$, such that ξ is maximal unless $L_0 = L_k$, and for $i = 1, \dots, r$: (i) $F_i \in B$, and (ii) L_{i-1} and L_i are non-reducible paths for F_i which are contained in opposite 1/2-segments in C_{F-i}. Let Ξ be the set of all such sequences ξ considered up to reversing and/or cyclical shifting (when $L_0 = L_k$). Then each non-reducible path (for some $F \in B$) belongs to a unique sequence $\xi \in \Xi$.

For $F \in B$ let $\pi_F = (P_1, \dots, P_{k(F)})$ be a sequence of all non- reducible paths in C_F. Define $\rho_F = (*_1, \dots, *_{k(F)})$ as follows:

(5.10) for $i = 1, \dots, k(F)$, if $\xi = (L_0, F_1, L_1, \dots, F_r, L_r)$ is the sequence in Ξ such that $P_i = L_j$ for some j then put $*_i := +$ if $F = F_{j+1}$ and put $*_i := -$ if $F = F_j$.

One can check that if for each $F \in B$, h'_F is the function on EG_F formed (as in (5.9)) from h_F by use of π_F and ρ_F defined as in (5.10), then each problem (h'_F, d_F) is solvable and the collection $\{h'_F : F \in B\}$ is admissible (i.e. $h'_F(e) + h'_{F'}(e) \leq 1$ for any $F, F' \in B$ and $e \in C_F \cup C_{F'}$). •

From (5.8) and Statement 5.5 it follows immediately that

(5.11) there is $F \in B$ such that $|\mathcal{R}(F)| \geq 3$, and $\mathcal{R}(F)$ contains three strong paths P_1, P_2, P_3 so that P_1 and P_2 belong to the same 1/2-segment in C_F that is opposite to the 1/2-segment containing P_3.

In particular, (5.11) shows that Q' as above cannot be of type Q'_1. Also if $|\mathcal{R}(F)| = 2$ then each of two (essential) vertices in Q' belonging to $\text{bd}(F)$ cannot be in Z^0. Hence, if Q' is of type Q'_2 or Q'_3 then $Z^0 = \emptyset$, and therefore Q is of type Q_2 or Q_3 as drawn in Fig. 5.3. Now we consider the other possible types for Q (or Q'). We use notations as in Fig. 5.3–5.5.

A. Q is of type Q_2. Let for definiteness the paths L_1, L_2, P in $\mathcal{R}(F(1))$ are strong; S_1, S_2, S are the 1/2-segments in $C_{F(1)}$ containing them, respectively; two of these segments are the same and opposite to the third one. From Lemma 5.4 it follows that $S \neq S_1, S_2$ (as, e.g., the 1-path connecting the vertices y, z in Fig. 5.3a contains an edge u with $h_{F(1)}(u) = 1$, by Lemma 5.4 applied to the 1-paths connecting x, y, z). Hence, $S_1 = S_2$ (and similarly for $F(3)$ if two of paths L_0, L_3, P belong to the same 1/2-segment for $C_{F(3)}$). For $i = 0, 1, 2, 3$ let h_i be the function on $EG_{F(i)}$ formed from $h_{F(i)}$ by use of π_i, ρ_i, where

$$\begin{aligned} \pi_0 &= (L_0, L_1) \text{ and } \rho_0 = (-, +), \\ \pi_1 &= (L_1, L_2, P) \text{ and } \rho_1 = (-, -, +), \\ \pi_2 &= (L_2, L_3) \text{ and } \rho_2 = (+, -), \\ \pi_3 &= (L_3, L_0) \text{ and } \rho_3 = (+, +, -), \end{aligned} \tag{5.12}$$

Fig. 5.3

Fig. 5.4

see Fig 5.4. One can check that each $(h_i, d_{F(i)})$ is solvable and the collection $\{h_1, h_2, h_3, h_4\}$ is admissible. Hence, $(G, U)^*$ has a half-integral solution.

B. Q is of type Q_3. Without loss of generality one may assume that P_1 is a non-reducible path for $F(1)$, and that P_1 and L_1 belong to the same 1/2-segment in $C_{F(1)}$. On the other hand, by Lemma 5.4 (applied to the 1-paths connecting the vertices

x, y, z as in Fig. 5.3b) the 1-path connecting y and z must contain an edge u with $h_{F(1)}(u) = 1$. Hence, P_1 and L_1 belong to different 1/2-segments; a contradiction.

C. Q' is of type Q'_4. Then $|\mathcal{R}(F)| = 3$ for all $F \in B$. Let $B = \{F(i) : i = 0, 1, 2, 3\}$, and let $P_{ij} = P_{ji}$ denote the maximal path in Q'_4 common for $\mathrm{bd}(F(i))$ and $\mathrm{bd}(F(j))$. Consider two cases.

Case 1. $Z^0 \neq \emptyset$. Let for definiteness $v \in Z^0$, where v is the vertex indicated in Fig. 5.2d. By Lemma 5.1, for $j = 0, 1, 2$ the paths $P_{j-1,j}$ and $P_{j,j+1}$ belong to the same 1/2-segment for $C_{F(j)}$ (indices are taken modulo 3); therefore, $P_{j,3}$ must belong to the opposite segment in $C_{F(j)}$. In particular, $|Z^0| = 1$, and Q is of type as in Fig. 5.5a.

Fig. 5.5

Next, the path P_{01} is strong, so by Lemma 5.4 (applied to the 1-paths connecting the vertices x, y, z as in Fig. 5.5a) the 1-path connecting y and z contains an edge u with $h_{F(3)}(u) = 1$. Hence, P_{30} and P_{31} belong to different 1/2-segments in $C_{F(3)}$, and similarly for P_{30}, P_{32} and for P_{31}, P_{32}. Then (by Lemma 5.3 and (5.8)) some of $P_{3,j}$, say P_{32}, is reducible for $F(3)$. For $i = 0, 1, 2, 3$ let h_i be the function on $EG_{F(i)}$ formed from $h_{F(i)}$ by use of π_i, ρ_i, where

$$
\begin{aligned}
&\pi_0 = (P_{01}, P_{02}, P_{03}) \ \text{ and } \ \rho_0 = (+, +, -),\\
&\pi_1 = (P_{10}, P_{12}, P_{13}) \ \text{ and } \ \rho_1 = (-, -, +),\\
&\pi_2 = (P_{20}, P_{21}, P_{23}) \ \text{ and } \ \rho_2 = (-, -, +),\\
&\pi_3 = (P_{30}, P_{31}, P_{32}) \ \text{ and } \ \rho_3 = (+, -, -),
\end{aligned}
\tag{5.13}
$$

see Fig. 5.5a.

Case 2. $Z^0 = \emptyset$. Then Q is of type as in Fig. 5.5b. Let for definiteness P_{20} and P_{21} belong to the same 1/2-segment in $C_{F(2)}$. Then Lemma 5.4 (for the 1-paths connecting x, y, z as in Fig. 5.5b) implies that P_{01} is not a strong path. Hence, all paths $P_{3,j}$, $j = 0, 1, 2$, are strong. Next, applying Lemma 5.4, we observe that the paths P_{30} and P_{32} belong to different 1/2-segments in $C_{F(3)}$, and similarly for P_{31} and P_{32}. For $i = 0, 1, 2, 3$ let h_i be the function on $EG_{F(i)}$ formed from $h_{F(i)}$ by use of π_i, ρ_i, where

$$
\begin{aligned}
(5.13) \qquad \pi_0 &= (P_{02}, P_{03}) \text{ and } \rho_0 = (+, -),\\
\pi_1 &= (P_{12}, P_{13}) \text{ and } \rho_1 = (+, -),\\
\pi_2 &= (P_{20}, P_{21}, P_{23}) \text{ and } \rho_2 = (-, -, +),\\
\pi_3 &= (P_{30}, P_{31}, P_{32}) \text{ and } \rho_3 = (+, +, -),
\end{aligned}
$$

see Fig. 5.5b.

A straightforward check-up shows that in both cases each problem $(h_i, d_{F(i)})$ is solvable and the collection $\{h_i\}$ is admissible, whence $(G, U)^*$ has a half-integral solution.

Thus, the case $|B| = 4$ is impossible.

6. EXCLUSION OF $|B| = 3$

We show that in this case either $(G, U)^*$ has a half-integral solution, or there is a reduction to case $|B| = 2$ or $|B| = 4$, whence Theorem 1 will follow. We need the following lemma which strengthens, in a sense, Lemma 5.4.

Lemma 6.1. *Let $F(0), F(1), F(2) \in B$ be distinct holes so that for $i = 0, 1, 2$:*

(a) *there is a 1-path $P_i = x_1^i x_2^i \dots x_{k(i)}^i$ in $C_{F(i)}$, and x_1^i coincides with $x_{k(i+1)}^{i+1}$;*

(b) *$C_{F(i)}$ and $C_{F(i+1)}$ have a common edge e_i one end of which is x_1^i, and $h_{F(i)}(e_i) = h_{F(i+1)}(e_i) = 1/2$ (indices are taken modulo 3).*

For $i = 0, 1, 2$ let $r(i)$ and $l(i)$ be the minimum and maximum indices so that for the edges $u_i = x_{r(i)}^i x_{r(i)+1}^i$ and $u_i' = x_{l(i)}^i x_{l(i)-1}^i$ one has $h_{F(i)}(u_i), h_{F(i)}(u_i') \in \{0, 1\}$. Then:

(i) *all the edges u_i, u_i' ($i = 0, 1, 2$) belong to the boundary of the same face of G in the region $\Omega \subset \mathbb{R}^2$ bounded by P_1, P_2, P_3; and*

(ii) $h_{F(i)}(u_i) = h_F(i)(u_i') = 1$.

Proof. Since each P_i is a 1-path, Ω contains no hole. Notice that there is no vertex $x \in VG$ in the interior of Ω (otherwise there would exist a fork $\tau = (e, x, e')$ with $f^e = f^{e'} = 0$, whence $\beta(\tau) = 1$). Hence, every edge lying in the interior of Ω connects vertices in $P_1 \cup P_2 \cup P_3$.

Suppose that (i) is not true for some $w \in \{u_i, u_i'\}$ and $w' \in \{u_{i'}, u_{i'}'\}$ for $i \neq i'$; let for definiteness $w = u_1$. Then one can yield from Lemma 5.4 that in the interior of Ω there is an edge e with ends $x = x_j^1$ and $y = x_{j'}^1$ for some $1 \leq j \leq r(1) < j' \leq k(1)$.

Consider the edge e' different from xx^1_{j+1} and such that $\tau = (e, x, e')$ is a fork, see Fig. 6.1.

Fig. 6.1

We show that $\beta(\tau) = 3/4$. Clearly e' does not lie in the interior of Ω (otherwise from $f^e = f^{e'} = 0$ it would follow that $\beta(\tau) = 1$). Hence, only the following two cases are possible.

(i) $j > 1$ and $e' = xx^1_{j-1}$. Then $f^{e'} = 1/2$ (as $j \le r(1)$), whence $\beta(\tau) = 3/4$ (in view of $f^e = 0$).

(ii) $j = 1$ and $e' = xx^2_{k(2)-1}$. Obviously, $E_{F(2)}(x) = \emptyset$. Therefore $f^e_{F(2)} = f^{e_1}_{F(2)} = 1/2$, and we again obtain $\beta(\tau) = 3/4$.

Let $E(x) = \{e, e', u, u'\}$ and $\tau' = (u, x, u')$, then $u' = xx^1_{j+1}$ and $\beta(\tau') = 3/4$ (by (4.5)). Denote $z := x^1_{j+1}$.

Consider a solution f' for $G_{\tau'}, c_{\tau',3/4}$ obtained from f as in Remark 4.5, and a 4f-metric m critical for τ' and induced by $\sigma : VG \to VH$ as in Theorem 3.11. Let $C' = C'_{F(1)}$ be the circuit for f' corresponding to $C_{F(1)}$. Since $\sigma(y) = \sigma(x)$ (as $(f')^e = 0$), $y, z \in C'$ and $\{\sigma(x), \sigma(z)\} = \{b_0, b_4\}$, we have $\sigma(F(1)) = \widetilde{J}$ (by (i),(iii)(a) in Theorem 3.11). This shows that the case (ii) as above is impossible (otherwise we would have $(f'_{F(2)})^{u'} > 0$, whence $\sigma(F(2)) = \widetilde{J}$). Hence, $j > 1$, which implies $C' = C_{F(1)}$ and $u' \in C'$.

Let for definiteness $\sigma(x) = b_0$. The vertices x, z, y occur in this order in C', and we have $\sigma(x) = \sigma(y) = b_0$ and $\sigma(z) = b_4$. Therefore, in view of Statement 4.7, $\sigma(x') = b_0$ for all vertices x' in the part of C' between x and y that does not contain z. But then the whole circuit $\widetilde{C}$ formed from C' by replacing the path $x^1_j x^1_{j+1} \ldots x^1_{j'}$ by the edge e is mapped by σ into the unique point b_0, which is impossible (since, e.g., $\widetilde{C}$ separates some holes).

From proved above it obviously follow that all the edges u_i, u'_i ($i = 0, 1, 2$) belong to the same face in Ω.

Finally, suppose that for some $u \in \{u_i, u'_i\}$, $h_{F(1)}(u) = 0$; let for definiteness $u = u_1$. Consider the fork $\tau = (u, x, e)$ belonging to the boundary of some face in Ω, where $x := x^1_{r(1)}$. One can see that either (i) $r_1 > 1$ and $e = x^1_{r(i-1)}x$, or (ii) $r(1) = 1$ and $e = xx^2_{k(2)-1}$. Also one can see that in both cases, $f^e = 1/2$, whence $\beta(\tau) = 3/4$. In case (i), we get a contradiction using arguments as above (with τ instead of τ'). In case (ii), e belongs to both circuits $C'_{F(1)}$ and $C'_{F(2)}$ (for f' defined as in Remark 4.5), which leads to a contradiction with (i),(iii)(a) in Theorem 3.11. •

Fig. 6.2

Now we begin to consider the case $|B| = 3$. Let $B = \{I, J, K\}$ and $\mathcal{H}_K = \{K, O\}$. The graph Q' (defined as in the previous section) can be only as drawn in Fig. 6.2a.

By (5.8) (for $F = I, J$), the paths P_1, P_2, P_3 are strong, P_1, P_2 are non-reducible for I, while P_2, P_3 are non-reducible for J. In particular, the graph Q is of form as in Fig. 6.2b. Let e_I be the first edge with $h_I(e_I) \in \{0, 1\}$ contained in the 1-path L_1 from x to y in C_I, and e_J be the first edge with $h_J(e_J) \in \{0, 1\}$ contained in the 1-path L_2 from x to z in C_J. Let u_I be the last edge with $h_I(u_I) \in \{0, 1\}$ contained in the 1-path L'_1 from x' to y' in C_I, and u_J be the first edge with $h_J(u_J) \in \{0, 1\}$ contained in the 1-path L'_2 from x' to z' in C_J, see Fig. 6.2b. By Lemma 6.1,

(6.1) $h_I(e_I) = h_I(u_I) = h_J(e_J) = h_J(u_J) = 1$; e_I and e_J are dually connected; u_I and u_J are dually connected.

Statement 6.2. *e_I and u_I belong to a tight cut δX_I for G_I, h_I, U_I (and similarly, e_J and u_J belong to a tight cut δX_J for G_J, h_J, U_J).*

Proof. Let for definiteness $L_1 = x_1 \dots x_k$ and $L'_1 = y_1 \dots y_r$, where $x_1 = x$ and $y_1 = x'$, and let $e_I = x_i x_{i+1}$ and $u_I = y_j y_{j+1}$. In view of (5.2) and the fact that the 1/2-segments containing P_1 and P_2 are opposite in C_I, every tight cut δX containing e_I meets L'_1 in some edge $w = y_{j'} y_{j'+1}$ with $h_I(w) \in \{0, 1\}$. Similarly, every tight cut δY containing u_I meets L_1 in some edge $z = x_{i'} x_{i'+1}$ with $h_I(z) \in \{0, 1\}$. Let δX (δY) be chosen so that j' is maximum (resp., i' is minimum). Suppose that $j' < j$; then $i' > i$. Consider the dual paths

$$D = (\widetilde{F}, e_1, F_1, \dots, e_{p-1}, F_{p-1}, e_p, \widetilde{F}) \quad \text{and} \quad D' = (\widetilde{F}, e'_1, F'_1, \dots, e'_{q-1}, F'_{q-1}, e'_q, \widetilde{F}),$$

where $\{e_1,\dots,e_p\}=\delta X$, $\{e'_1,\dots,e'_q\}=\delta Y$, $e_1=e_I$, $e_p=w$, $e'_1=z$, $e'_q=u_I$, and $\widetilde{F}$ is the face in G_I bounded by C_I. Let $e_s,e'_t,e'_{t+1},e_{s+1}$ lie in $\mathrm{bd}(I)$. Using arguments similar to those in the proof of Lemma 5.3 and taking into account the choice of i',j', one can show that D and D' have no common face different of I and $\widetilde{F}$. This implies that $e_s,e'_t,e'_{t+1},e_{s+1}$ occur in this order in $\mathrm{bd}(I)$. Then the sets

$$\delta X':=\{e_1,\dots,e_s,e'_{t+1},\dots,e'_q\}\quad\text{and}\quad\delta Y':=\{e'_1,\dots,e'_t,e_{s+1},\dots,e_p\}$$

are tight cuts. A contradiction with the maximality of i'. •

From Statement 6.2 and Lemma 6.1 it follows that $\delta X_I\cup\delta X_J$ forms a strong cut δZ in $G_I\cup G_J$ (with all-unit capacities of the edges), that is,

$$|\delta Z|=d_I(Z)+d_J(Z).$$

This means that for *any* solution f' of $(G,U)^*$ the edges in δZ must be saturated by the flow $f'_I+f'_J$, therefore,

(6.2) for any solution f' of $(G,U)^*$, I and J belong to some bunch B'.

Now we consider the graph G_K. Let $\mathcal{R}$ be the set of (simple) tight (for h_K,U_K) cuts in G_K that meet twice the edge-set in $P_1\cup P_3$. Suppose that some of P_1 and P_3, say P_1, has the property that no cut δX in $\mathcal{R}$ meets twice P_1. Then define the function h'_K on EG_K by

$$\begin{aligned} h'_K(e) &:= 0 \quad \text{if } e\in P_1,\\ &:= 1 \quad \text{otherwise,}\end{aligned}$$

and define h'_I,h'_J on EG_I,EG_J, respectively, by

$$\begin{aligned} h'_I(e) &:= 0 \quad \text{if } e\in P_2,\\ &:= 1 \quad \text{otherwise;}\\ h'_J(e) &:= 0 \quad \text{if } e\in P_3,\\ &:= 1 \quad \text{otherwise.}\end{aligned}$$

Then each (h'_F,d_F), $F\in\{I,J,K\}$, is solvable, and the collection $\{h'_I,h'_J,h'_K\}$ is admissible. Therefore, $(G,U)^*$ has a half-integral solution.

Thus, there is a cut $\delta X\in\mathcal{R}$ that meets twice P_1, and similarly, there is a cut $\delta X'$ that meets twice P_3. Let L (L') be the 1- path in C_K from z to y (resp., from z' to y'), and let $\widetilde{F}$ be the face in G_K bounded by C_K, see Fig. 6.3a.

Next, denote by Q the set of edges w in $L\cup L'$ with $h_K(w)\in\{0,1\}$. Let a (b) be the first edge in L (resp., in L') belonging to Q. By Lemma 6.1, (6.3) $h_K(a) = h_K(b) = 1$; a is dually connected with e_I and e_J; b is dually connected with u_I and u_J.

Fig. 6.3

Let $\mathcal{A}$ be the set of all tight cuts in G_K that meet Q. From arguments as in the proof of Lemma 5.3 it follows that

(6.4) for any $\delta Y \in \mathcal{A}$ and $\delta Z \in \mathcal{R}$ their corresponding dual paths D_Y and D_Z in G_K have no common face different from $\widetilde{F}, K, O$, and if they have a common face $F \in \{K, O\}$ then they are crossing at this face.

Statement 6.3. *There exists $\delta Z \in \mathcal{A}$ that meet both* $\mathrm{bd}(K)$ *and* $\mathrm{bd}(O)$ *and contains the edges a and b.*

Proof. Suppose that some of $\delta X, \delta X'$, say δX, meets only one of $\mathrm{bd}(K), \mathrm{bd}(O)$, say $\mathrm{bd}(K)$. From (6.4) it follows that each cut in $\mathcal{A}$ meets only O, which implies that $\delta X'$ meets $\mathrm{bd}(K)$, see Fig. 6.3a. But then for at least one $\widetilde{L} \in \{L, L'\}$ the dual path D_Z for any cut $\delta Z \in \mathcal{A}$ which meets $\widetilde{L}$ must have a common face $F \neq \widetilde{F}$ with D_X or a common face $F \neq \widetilde{F}, K, O$ with $\delta X'$; a contradiction with (6.4).

Hence, each $\delta X, \delta X'$ meets both $\mathrm{bd}(K)$ and $\mathrm{bd}(O)$. Applying (6.4), it is not difficult to show that every cut in $\mathcal{A}$ meets $L, L', \mathrm{bd}(K), \mathrm{bd}(O)$. Now the statement is proved by use of arguments as in the proof of Statement 6.2. •

From (6.1),(6.3) and Statements 6.2 and 6.3 it follows that

(6.5) if f' is an arbitrary solution of $(G, U)^*$ that all edges in the set δX_I are saturated by the flow f'_I, all edges in δX_J are saturated by f'_J, and all edges in δZ are saturated by $f'_K + f'_O$.

Now using (6.5) it is easy to show that

(6.6) for any solution f' for $(G, U)^*$ the circuits C_{IJ} and C_{JI} are neighbouring, C_{IJ} does not separate J, K, O, and C_{JI} does not separate I, K, O.

Return to the flow f, and consider the bunch $B' = \{K, O\}$. Apply the operation of "balancing" to C_{KO} and C_{OK} as in the proof of Lemma 2.2. From the proof of Lemma 2.2 one can see that as a result we get a solution f' for $(G,U)^*$ and a bunch $\widetilde{B}$ satisfying the statement of this lemma and such that $K, O \in \widetilde{B}$. Two cases are possible.

(i) $|\widetilde{B}| = 2$. Then $(G,U)^*$ has a half-integral solution according to proved in Section 4.

(ii) $|\widetilde{B}| > 2$. Then, in view of (6.6), $\widetilde{B} = \{I, J, K, O\}$, whence $(G,U)^*$ has a half-integral solution according to proved in Section 5.

This completes the proof of Theorem 1. • • •

REFERENCES

[Iri] M. Iri, On an extension of the maximum-flow minimum-cut theorem to multicommodity flows, *J. Oper. Res. Soc. Japan* **13** (1070/71) 129-135.

[Ka1] A.V. Karzanov, Paths and metrics in a planar graph with three or more holes, Part I: Metrics, *Research Report RR 816-M-* (IMAG ARTEMIS, Grenoble, 1990) 21p, to appear in *J. Combinatorial Theory.*

[Ka2] A.V. Karzanov, Paths and metrics in a planar graph with three or more holes, Part II: Paths, *Research Report RR 817-M-* (IMAG ARTEMIS, Grenoble, 1990) 21p, to appear in *J. Combinatorial Theory.*

[Ok] H. Okamura, Multicommodity flows in graphs, *Discrete Applied Mathematics* **6** (1983) 55-62.

[OkS] H. Okamura and P.D. Seymour, Multicommodity flows in planar graphs, *Journal of Combinatorial Theory (B)* **31** (1981) 75-81.

[KeO] O. Kenji and K. Osamu, On feasibility conditions on multicommodity flow in networks, *IEEE Trans. Circuit Theory* **CT-18** (1971) 425-429.