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# Paths and Metrics in a Planar Graph with Three or More Holes. I. Metrics*

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Let $G=(VG, EG)$ be a bipartite planar graph embedded in the euclidean plane and $\mathscr{H}$ be a subset of its faces (*holes*). A. Schrijver proved that if $|\mathscr{H}|=2$ then there exists a collection $\mathscr{C}=\{m_1, ..., m_k\}$ of cut metrics on $VG$ such that: (i) $m_1(e)+\cdots+m_k(e)\leqslant 1$ for any $e\in EG$; and (ii) for any vertices $s$ and $t$ in the boundary of a hole, the value $m_1(s,t)+\cdots+m_k(s,t)$ is equal to the distance between $s$ and $t$. This is, in general, not true for $|\mathscr{H}|=3$.

In the present paper one proves that: (*) for $|\mathscr{H}|=3$, (i)–(ii) hold for some $\mathscr{C}$ consisting of cut metrics and 2, 3-metrics (metrics induced by the graph $K_{2,3}$); and (**) for $|\mathscr{H}|=4$, (i)–(ii) hold for some $\mathscr{C}$ consisting of metrics induced by planar graphs with at most four faces.

Using (*), in the sequel to the present paper (Part II) we give a criterion of the existence of edge-disjoint paths connecting certain vertices in a planar graph with three holes, provided that the so-called "parity condition" holds. This extends, in a sense, Okamura's theorem on edge-disjoint paths in planar graphs with two holes. © 1994 Academic Press, Inc.

## 1. INTRODUCTION

Throughout, we deal with a connected undirected graph $G$ embedded in the euclidean plane $\mathbb{R}^2$. $VG$ is the vertex set and $EG$ is the edge set of $G$ (multiple edges are admitted). The set of faces of $G$ is denoted by $\mathscr{F}=\mathscr{F}_G$. A subset $\mathscr{H}\subseteq\mathscr{F}$ of faces in $G$ is distinguished; a face $F\in\mathscr{H}$ is called a *hole*.

A. Schrijver proved the following theorem on packings of cuts.

SCHRIJVER'S THEOREM [12]. *Let $G$ be bipartite, and let $|\mathscr{H}|\leqslant 2$. Then there exist disjoint cuts $\delta X_1, ..., \delta X_k$ in $G$ such that*

$$|\{i \mid \delta X_i \text{ separates } s \text{ and } t\}| = \operatorname{dist}^G(s,t) \qquad \text{for all} \quad s,t\in V(I),\quad I\in\mathscr{H}. \qquad (1)$$

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[For $X \subseteq VG$, $\delta X = \delta^G X$ denotes the set of edges of $G$ with one end in $X$ and the other in $VG - X$; a nonempty set $\delta X$ is called a *cut* in $G$; $\delta X$ *separates* vertices $x$ and $y$ if $X$ contains exactly one of $x$ and $y$; $\text{dist}^G(x, y)$ denotes the distance in $G$ between vertices $x$ and $y$; $V(F)$ is the set of vertices of $G$ contained in the boundary $\text{bd}(F)$ of a face $F$.] In [4] another proof of this theorem was given which provides a strongly polynomial algorithm (supposing that $G$ is edge-weighted).

Schrijver's theorem does not remain true for $|\mathscr{H}| = 3$; e.g., consider $G$ to be the complete bipartite graph $K_{2,3}$, see Fig. 1. Nevertheless, Theorem 1 can be extended, in a sense, to $|\mathscr{H}| = 3$ if one adds to the cuts the set of so-called 2, 3-metrics.

Here and later on by a *metric* on a set $V$ we mean a real-valued function $m$ on $V \times V$ satisfying $m(x, x) = 0$, $m(x, y) = m(y, x)$, and $m(x, y) + m(y, z) \geqslant m(x, z)$ for all $x, y, z \in V$. We say that $m$ is *induced* by $(\Gamma, \sigma)$, where $\Gamma$ is a graph and $\sigma$ is a mapping of $V$ into $V\Gamma$, if $m(x, y)$ is equal to $\text{dist}^\Gamma(\sigma(x), \sigma(y))$ for all $x, y \in V$; when it leads to no confusion, we may say that $m$ is induced by $\Gamma$ or $m$ is induced by $\sigma$. If $\sigma(V) = V\Gamma$ and $\Gamma$ is $K_2$ ($K_{2,3}$, respectively) then $m$ is said to be a *cut metric* (a 2, 3-*metric*, respectively). Clearly there is a natural one-to-one correspondence between the cuts in $G$ and the cut metrics on $VG$. In its turn, a 2, 3-metric $m$ on $VG$ is determined uniquely by an ordered partition of $VG$ into five nonempty subsets, say $(T_1, T_2, S_1, S_2, S_3)$, considered up to a permutation of $T_1, T_2$ and a permutation of $S_1, S_2, S_3$.

It is easy to see that if $m_1, \ldots, m_k$ are metrics on $VG$ such that

$$m_1(e) + \cdots + m_k(e) \leqslant 1 \qquad \text{for all} \quad e \in EG, \tag{2}$$

then $m_1(x, y) + \cdots + m_k(x, y) \leqslant \text{dist}^G(x, y)$ holds for any $x, y \in VG$ (for a metric $m$ and an edge $e$ with ends $u$ and $v$, $m(e)$ stands for $m(u, v)$). In metrics terms, Schrijver's theorem asserts that if $G$ is bipartite and $|\mathscr{H}| = 2$, then there exist cut metrics $m_1, \ldots, m_k$ on $VG$ satisfying (2) and

$$m_1(s, t) + \cdots + m_k(s, t) = \text{dist}^G(s, t) \qquad \text{for all} \quad I \in \mathscr{H} \quad \text{and} \quad s, t \in V(I). \tag{3}$$

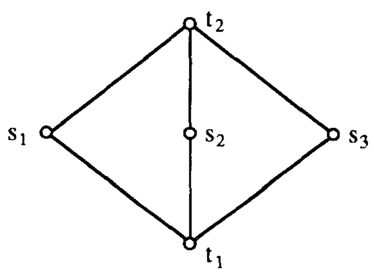


FIGURE 1

In the present paper we prove the following theorem.

THEOREM 1. *Let $G$ be bipartite, and let $|\mathscr{H}| = 3$. Then there exist $m_1, ..., m_k$ satisfying* (2) *and* (3), *where each $m_i$ is a cut metric or a* 2, 3-*metric on $VG$.*

Figure 2 illustrates two instances of bipartite graphs with four holes for which (2)–(3) cannot be satisfied by taking only cut metrics and 2, 3-metrics. The natural question arises: given $n$, is it true that there exists a *finite* set $\mathscr{G}$ of graphs providing the property that, for any $(G, \mathscr{H})$ with $G$ bipartite and $|\mathscr{H}| = n$, (2)–(3) hold with all $m_i$'s induced by only graphs from $\mathscr{G}$? It turns out that even for $n = 4$ the answer is negative (moreover, it remains negative even for the fractional relaxation of the problem with $n = 4$); we show this in Section 4. Nevertheless, the following is true.

THEOREM 2. *Let $G$ be bipartite, and let $|\mathscr{H}| = 4$. Then there exist metrics $m_1, ..., m_k$ on $VG$ satisfying* (2) *and* (3), *where each $m_i$ is a cut metric or a* 2, 3-*metric or a metric induced by a bipartite planar graph $\Gamma$ with four faces.*

In contrast to this result, if $n = 5$ then (2)–(3) cannot be, in general, satisfied by using only planar graphs $\Gamma$ with at most five faces, as shown in Section 4.

Theorems 1 and 2 will be proved in Section 2. In Section 3 Theorem 1 is extended by showing that 2, 3-metrics in this theorem can be chosen to correspond, in a sense, to the topological structure of the space $\mathbb{R} - \bigcup (I \in \mathscr{H})$ (Theorem (3.1)). Basing on this strengthening and using a relationship between packings of metrics and multicommodity flows, in the sequel to this paper (Part II) we obtain a criterion for the existence of edge-disjoint paths connecting certain pairs of vertices in a planar graph (Theorems 3 and 4 below).

To formulate these results, consider a family $U = \{\{s_1, t_1\}, ..., \{s_r, t_r\}\}$ of pairs of vertices in $G$. Let $k$ be an integer $\geqslant 1$.

PROBLEM. $(G, U, k)$: find paths $P_1^1, ..., P_1^k, P_2^1, ..., P_2^k, ..., P_r^1, ..., P_r^k$ in $G$ such that each edge of $G$ occurs at most $k$ times in these paths, and $P_i^j$ connects $s_i$ and $t_i$.

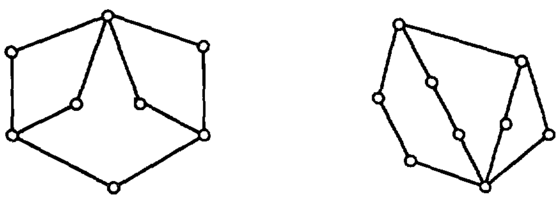

FIGURE 2

When $k=1$, $(G, U, k)$ turns into the problem of finding edge-disjoint paths $P_1, ..., P_r$ in $G$ such that $P_i$ connects $s_i$ and $t_i$. An obvious necessary condition for the solvability of $(G, U, k)$ (that is, for the existence of required paths) is the *cut condition*:

$$\text{each cut } \delta X \text{ of } G \text{ separates at most } |\delta X| \text{ pairs in } U. \tag{4}$$

We say that $G$ and $U$ satisfy the *parity condition* if:

$$|\delta X| + |\{i=1, ..., r \mid \delta X \text{ separates } s_i \text{ and } t_i\}| \text{ is even for each } \quad X \subset VG. \tag{5}$$

Let $\mathscr{H}$ be a set of faces in $G$ such that for each $i=1, ..., r$ there exists an $I \in \mathscr{H}$ such that $s_i, t_i \in V(I)$. Okamura proved the following theorem.

OKAMURA'S THEOREM [9]. *If* $|\mathscr{H}|=2$ *and if* (4) *and* (5) *hold, then the problem* $(G, U, 1)$ *has a solution.*

[An analogous theorem for $|\mathscr{H}|=1$ was proved by Okamura and Seymour [10].] In particular, from Okamura's theorem it follows that: if $|\mathscr{H}|=2$, (5) holds and $(G, U, k)$ is solvable for some $k$, then $(G, U, 1)$ is solvable as well. An unexpected fact is that the same remains true for $|\mathscr{H}|=3$.

THEOREM 3. *If* $|\mathscr{H}|=3$, (5) *holds and* $(G, U, k)$ *has a solution for some* $k$, *then* $(G, U, 1)$ *has a solution as well.*

Note that in case $|\mathscr{H}|=3$ the cut condition (4) is, in general, not sufficient for the solvability of $(G, U, k)$ with any fixed $k$ (e.g., consider $G := K_{2,3}$ and $U := \{\{s_1, s_2\}, \{s_2, s_3\}, \{s_3, s_1\}, \{t_1, t_2\}\}$, see Fig. 1). A solvability criterion for it involves an additional condition on 2, 3-metrics, as follows.

THEOREM 4. *Let* $|\mathscr{H}|=3$. *The problem* $(G, U, k)$ *has a solution for some* $k$ *if and only if* (4) *and*

$$\sum_{e \in EG} m(e) \geq \sum_{i=1}^{r} m(s_i, t_i) \qquad \text{for all 2, 3-metrics } m \text{ on } VG \tag{6}$$

*hold.*

In conclusion of this section let us mention analogues of Theorems 1, 3, and 4 for arbitrary graphs $G$. Let $T(U)$ be the set of distinct vertices among $s_i$'s and $t_j$'s.

(i) If $|T(U)| \leqslant 5$, (5) holds and $(G, U, k)$ has a solution for some $k$, then $(G, U, 1)$ has a solution as well; this fact was proved for $|T(U)| \leqslant 4$ in [6, 13] (see also [8]) and for $|T(U)| = 5$ in [3].

(ii) If $|T(U)| \leqslant 4$, then $(G, U, k)$ has a solution for some $k$ if and only if (4) holds [11]; if $|T(U)| = 5$, then $(G, U, k)$ has a solution for some $k$ if and only if (4) and (6) hold [3].

(iii) Let $G$ be a connected bipartite graph, and let $T \subseteq VG$. If $|T| \leqslant 4$, then there exist disjoint cuts $\delta X_1, ..., \delta X_N$ in $G$ such that the equality in (1) holds for all $s, t \in T$ [2]. If $|T| = 5$, then there exist metrics $m_1, ..., m_k$ on $VG$, where each $m_i$ is a cut metric or a 2, 3-metric, satisfying (2) and such that the equality in (3) holds for all $s, t \in T$ [5].

It is known that the statements in (i)–(iii) do not remain true for $|T(U)| = 6$ and for $|T| = 6$.

## 2. Proofs of Theorems 1 and 2

Below we do not differ a graph $G$ (a vertex, an edge of $G$) and its image in the plane. A face is considered as an open region in $\mathbb{R}^2$, that is, it is identified with its interior. A face in $\mathscr{F} - \mathscr{H}$ is called *intermediate*. Denote by $W = W(\mathscr{H})$ the set of pairs $\{s, t\}$ of vertices in $G$ such that $s, t \in V(I)$, $I \in \mathscr{H}$. An $s-t$ path with $\{s, t\} \in W$ is called a *W-path*. The distance function $\text{dist}^G$ is denoted by $d$.

An *$x-y$ path* in $G$ is meant to be a sequence $P = (x = x_0, e_1, x_1, ..., e_k, x_k = y)$, where $e_i$ is an edge of $G$ with end vertices $x_{i-1}$ and $x_i$; if $x = y$, $P$ is a *circuit*. The *length* $k$ of $P$ is denoted by $|P|$. When it does not lead to confusion we identify a path (circuit) with its image in the plane. The boundary $\text{bd}(F)$ of a face $F$ will be often considered as a (possibly not simple) circuit oriented clockwise from a point in $F$.

We use some ideas and techniques developed in [12] for case $|\mathscr{H}| = 2$. We proceed by induction on

$$\omega(G) := \sum (2^{|\text{bd}(F)|} \mid F \in \mathscr{F}_G), \tag{7}$$

considering the set of pairs $(G, \mathscr{H})$ with $G$ bipartite and $|\mathscr{H}| \leqslant 4$. First we eliminate some cases for which one can fulfil a simple reduction of $G$ to a bipartite graph $G'$ with $\omega(G') < \omega(G)$.

(i) Suppose that $\text{bd}(F) = (x_0, e_1, x_1, ..., e_{2k}, x_{2k} = x_0)$ and $k \geqslant 3$ for some intermediate face $F$. Then we subdivide $F$ into quadrangles as done in [12]. Add into $F$ new vertices $y_1, ..., y_{k-2}$ and edges $\{x_0, y_1\}$, $\{y_1, y_2\}$, $\{y_2, y_3\}$, ..., $\{y_{k-3}, y_{k-2}\}$ and $\{y_{k-2}, x_{k-1}\}$, $\{y_{k-2}, x_{k+1}\}$. Such a transformation preserves the distances for all the old vertices of $G$.

Furthermore, the value in (7) becomes smaller, whence the result follows by induction.

(ii) Suppose that $\delta X$ is a *reducible* cut in $G$. This means that

$$\begin{aligned} d(s,t) &= \mathrm{dist}^{G'}(s',t')+1 \\ &\qquad \text{if } \{s,t\}\in W \quad \text{and} \quad \delta X \text{ separates } s \text{ and } t, \\ d(s,t) &= \mathrm{dist}^{G'}(s',t') \\ &\qquad \text{for the other } \{s,t\} \text{ in } W, \end{aligned} \tag{8}$$

where $G'$ arises from $G$ by contracting each edge $e$ in $\delta X$ (that is, by identifying the ends of $e$ and deleting the loop), and $x'$ denotes the image of $x \in VG$ in $G'$. Clearly $G'$ is bipartite and $\omega(G') < \omega(G)$. So, by induction, there are $m'_1, \ldots, m'_k$ on $VG'$ (where each $m'_i$ is a cut metric or a 2, 3-metric) satisfying (2)–(3) for $G'$ with the set $\mathscr{H}'$ of holes corresponding to $\mathscr{H}$. Put $m_i(x,y) := m'_i(x',y')$ for $x, y \in VG$, $i = 1, \ldots, k$. Let $m_{k+1}$ be the cut metric determined by $\delta X$. From (8) it follows that $m_1, \ldots, m_{k+1}$ give a solution for $G$ and $\mathscr{H}$.

(iii) Let $x$ and $y$ be vertices in the boundary of an intermediate face $F$, and let $d(x,y)$ be even. Denote by $G'$ the graph obtained from $G$ by identifying $x$ and $y$. Suppose that $d(s,t) = \mathrm{dist}^{G'}(s',t')$ for all $\{s,t\} \in W$. Then the result for $G$, $\mathscr{H}$ obviously follows by induction.

(iv) Suppose that $G$ is not 2-connected. Then it can be split at some vertex $v$, forming two nontrivial graphs $G_i$, $i = 1, 2$. Choose properly holes in each $G_i$. The result easily follows by induction, taking into account that for $u \in VG_1$ and $w \in VG_2$, $d(u,w) = \mathrm{dist}^{G_1}(u,v) + \mathrm{dist}^{G_2}(v,w)$.

(v) Suppose that $e$ and $e'$ are parallel edges in $G$. Delete $e$ and choose properly holes in the obtained graph. This operation does not change any distances in the graph, and the result follows by induction.

In view of (i)–(v), one may assume that:

(9) $G$ has no multiple edges;

(10) each intermediate face $F$ is a quadrangle, that is, $|\mathrm{bd}(F)| = 4$;

(11) the circuit $\mathrm{bd}(F)$ is simple for each face $F$;

(12) $G$ has no reducible cuts;

(13) for each intermediate face $F$ with $\mathrm{bd}(F) = x_0x_1x_2x_3x_0$, there is a shortest $W$-path containing the vertices $x_0$ and $x_2$ (therefore, there is a shortest $W$-path containing the edges $x_0x_1$ and $x_1x_2$).

[An edge of $G$ with ends $x$ and $y$ is denoted by $xy$; a path $(x_0, e_1, x_1, \ldots, e_k, x_k)$ is denoted by $x_0x_1 \cdots x_k$.] The following lemma is the core of the proof.

(2.1) LEMMA. *$G$ has no intermediate faces, that is, $\mathscr{F} = \mathscr{H}$.*

*Proof.* Suppose that this is false. Following [12], by a *dual path* we mean a minimal sequence

$$D = (F_0, e_1, F_1, e_2, ..., F_{k-1}, e_k, F_k)$$

such that: (a) $F_{i-1}$ and $F_i$ are distinct faces in $G$, and $e_i$ is a common edge in $\mathrm{bd}(F_{i-1})$ and $\mathrm{bd}(F_i)$; (b) for $i = 1, ..., k-1$, the face $F_i$ is intermediate, and $e_i$ and $e_{i+1}$ are opposite edges in $\mathrm{bd}(F_i)$; (c) either $F_0, F_k \in \mathscr{H}$, or $F_0 = F_k \in \mathscr{F} - \mathscr{H}$ and $e_1, e_k$ are opposite edges in $\mathrm{bd}(F_0)$. For $i = 1, ..., k$, let $\mathrm{bd}(F_i) = x_i y_i y_{i+1} x_{i+1} x_i$ and $e_i = x_i y_i$. The path $R(D) := x_1 x_2 \cdots x_k$ (respectively, $L(D) := y_1 y_2 \cdots y_k$) is called the *right* (respectively, *left*) *boundary* of $D$. It was proved in [12] that

$$d(x_i, x_j) = d(y_i, y_j) \qquad \text{for all} \quad 1 \leqslant i \leqslant j \leqslant k \tag{14}$$

(note that the proof of (14) essentially uses (13) and it, in fact, does not depend on the number of holes); moreover, (12) and (14) easily imply the following properties:

(15) $D$ is not self-intersecting, that is, $F_i \neq F_j$ for $i \neq j$; in particular, the case $F_0 = F_k$ is impossible;

(16) each shortest $W$-path has at most one common edge with $D$.

For a path $P = v_0 v_1 \cdots v_k$ and $0 \leqslant i \leqslant j \leqslant k$ let $P(v_i, v_j)$ denote the part $v_i \cdots v_j$ of $P$ from $v_i$ to $v_j$. For a path $Q = u_0 u_1 \cdots u_{k'}$ with $u_0 = v_k$, $P \cdot Q$ stands for the concatenated path $v_0 v_1 \cdots v_k u_1 \cdots u_{k'}$. For a simple path $P$ with both ends in the boundary of a hole $I$, denote by $\mathscr{R}(P) = \mathscr{R}(P, I)$ the pair $\{\Omega_1, \Omega_2\}$ of closed regions whose union is $\mathbb{R}^2 - I$ and whose intersection is $P$.

CLAIM 1. *Let $I \in \mathscr{H}$, $F \in \mathscr{F} - \mathscr{H}$, $\mathrm{bd}(F) = xyx'y'x$, and let the edge $xy$ lie in $\mathrm{bd}(I)$. Then the edge $yx'$ is not in $\mathrm{bd}(I)$.*

*Proof.* Suppose that this is not so. Let $P$ be a shortest $W$-path containing $y, x', y'$, say $P = p \cdots yx'y' \cdots q$ ($P$ exists by (13)). Since $P$ is shortest and $xy, yx' \in \mathrm{bd}(I)$, one must be $p = y$, whence $p, q \in V(I)$. For $\mathscr{R}(P) = \{\Omega_1, \Omega_2\}$, let $\mathscr{H}_i$ be the set of holes of $G$ contained in $\Omega_i$. Since $|\mathscr{H}| \leqslant 4$, one may assume that $|\mathscr{H}_1| \leqslant 1$. Next, one may assume that $F$ lies in $\Omega_1$ (otherwise we could take the shortest path $pxy' \cdots q$ instead of $P$). For $P' := P(x', q)$, let $G_1$ be the subgraph of $G$ lying in $\Omega_1$, and let $G_2$ be the subgraph of $G$ such that $G_1 \cup G_2 = G$ and $G \cap G_2 = P'$. Let $I'$ be the face of $G_1$ involving $I$; then $P$ is a part of $\mathrm{bd}(I')$. By Schrijver's theorem, there exist disjoint cuts $\delta X_1, ..., \delta X_k$ in $G_1$ satisfying (1) for $G_1$ and $\mathscr{H}' :=$

$\mathscr{H}_1 \cup \{I'\}$. Obviously, one of these, say $\delta X_1$, contains the edge $yx'$; let $y \in X_1$. We know that $\delta X_1$ is reducible for $G_1$ and $\mathscr{H}'$, $P$ is shortest and $p, q \in V(I')$, hence (by (16)), $P'$ has no common edge with $\delta X_1$. So $\delta X_1$ is a cut in $G$, and $X_1 \cap VG_2 = \emptyset$.

We assert that $\delta X_1$ is reducible for $G$ and $\mathscr{H}$. To see this, consider $\{s, t\} \in W$ and an $s-t$ path $L$. We have to show that

$$|L - \delta X_1| \geqslant d(s, t) - \alpha(L), \tag{17}$$

where $\alpha(L) := 1$ if $\delta X_1$ separates $s$ and $t$, and 0 otherwise. Let us split $L$ as $L_1 \cdot L_2 \cdot \cdots \cdot L_r$ so that $r = r(L)$ is minimum, provided $L_i$ is a $u_i - u_{i+1}$ path contained in some $G_j$ (then $u_2, ..., u_r$ are vertices in $P'$). We use induction on $r(L)$. Inequality (17) is obvious if $L$ is a path in $G_2$ (since $L \cap \delta X_1 = \emptyset$) or $L$ is a path in $G_1$ (since $\delta X_1$ is reducible for $G_1$ and $\mathscr{H}'$). Let $r = 2$. One may assume that $L_1$ is in $G_1$. Then $L_2 \cap \delta X_1 = \emptyset$, whence

$$|L - \delta X_1| = |L_1 - \delta X_1| + |L_2| \geqslant d(u_1, u_2) - \alpha(L_1) + d(u_2, u_3) \geqslant d(s, t) - \alpha(L)$$

(taking into account that $\delta X_1$ is reducible for $G_1$ and $\mathscr{H}'$). Finally, for $r \geqslant 3$, one can replace $L_2$ by the $u_2 - u_3$ path $Q$ that is a part of $P'$ (or of the reverse path $P'^{-1}$). Observe that $|Q| \leqslant |L_2 - \delta X_1|$ (taking into account that $u_2, u_3 \in V(I')$ and $Q$ is shortest). Now for the path $L' := L_1 \cdot Q \cdot L_3 \cdot \cdots \cdot L_r$ we have $|L' - \delta X_1| \leqslant |L - \delta X_1|$, $\alpha(L') = \alpha(L)$ and $r(L') < r(L)$, whence (17) follows by induction. So $\delta X_1$ is reducible for $G$ and $\mathscr{H}$, contradicting (12). ∎

CLAIM 2. *Let $P$ be a shortest $s-t$ path with $s, t \in V(I)$, $I \in \mathscr{H}$, let $\mathscr{R}(P, I) = \{\Omega_1, \Omega_2\}$, and let $\Omega_1$ contain no hole. Then $P$ is contained in the boundary of $I$.*

*Proof.* Suppose that this is not so. Let $Q$ be the $s-t$ path in the boundary of $I$ such that $P$ and $Q$ form the boundary of $\Omega_1$. Denote by $n(P)$ the number of faces of $G$ that are in $\Omega_1$. Since $P \neq Q$, $n(P) > 0$. One may assume that $P$ is chosen so that $n(P)$ is minimum (subject to $n(P) > 0$) and $P$ has no inner vertex belonging to $Q$.

Take an edge $e = uv$ in $Q$. Since $\Omega_1$ contains no hole, $e$ belongs to the boundary of an intermediate face $F$. Furthermore, $F$ lies in $\Omega_1$. Let $\mathrm{bd}(F) = uvxyu$. Consider the dual path $D = (F_0, e_1, F_1, ..., e_k, F_k)$ traversing the edges $vx$ and $uy$. We know that $F_0 \neq F_k$ (by (15)), $\Omega_1$ has no hole, and $D$ can meet $P$ at most in one edge (by (16)). Hence, one of the end faces of $D$, say $F_0$, is $I$, and all faces of $D$ between $I$ and $F$ are in $\Omega_1$. Let for definiteness $D$ traverses $vx$ earlier than $uy$. One can see that $v$ belongs to the right boundary $R(D) =: y_1 y_2 \cdots y_k$, while $x$ belongs to the left boundary $L(D) =: x_1 x_2 \cdots x_k$ of $D$, and that $y_1$ and $v$ are inner vertices in

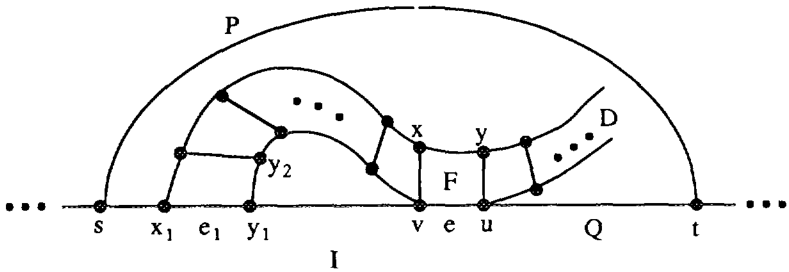


FIGURE 3

$Q(x_1, u)$, see Fig. 3. This implies that the part $R'$ of $R(D)$ from $y_1$ to $v$ does not meet $P$. Now choose a shortest $W$-path passing $x_1, y_1, y_2$, say $\tilde{P} = p \cdots x_1 y_1 y_2 \cdots q$. Since $\tilde{P}$ and $D$ have no common edge except $x_1 y_1$ (by (16)), the path $P' := \tilde{P}(y_1, q)$ lies in the region $\Omega' \subset \Omega_1$ bounded by $Q(y_1, v)$ and $R'$. Therefore, $q \in V(I)$, and $\Omega'' \subseteq \Omega' \subset \Omega_1$ for some $\Omega'' \in \mathscr{R}(P', I)$ (so $\Omega''$ contains no hole). In view of the minimality of $P$, the path $P'$ must coincide with the part of $Q$ from $y_1$ to $q$. Thus, both edges $x_1 y_1$ and $y_1 y_2$ are in $\mathrm{bd}(I)$—a contradiction with Claim 1 (for $I$ and $F_1$). ▮

Consider an intermediate face $F$; let $\mathrm{bd}(F) = uvxyu$. Choose a shortest $W$-path passing $u, v, x$, say $P = s \cdots uvx \cdots t$, and a shortest $W$-path passing $v, u, y$, say $Q = p \cdots vuy \cdots q$. Let $s, t \in V(I)$ and $p, q \in V(I')$, where $I, I' \in \mathscr{H}$. Denote $P_1 := P(s, u)$, $P_2 := P(x, t)$, $Q_1 := Q(p, v)$, $Q_2 := Q(y, q)$.

CLAIM 3. (i) *The paths* $P_1, P_2, Q_1, Q_2$ *are disjoint.* (ii) $I = I'$.

*Proof.* Clearly $P_1 \cap P_2 = Q_1 \cap Q_2 = \varnothing$. Suppose that for some $i, j \in \{1, 2\}$, $P_i$ and $Q_j$ have a common vertex $z$; let for definiteness $i = j = 1$ (if, e.g., $i = 1, j = 2$, one should consider the path $P_1 \cdot uyx \cdot P_2$ instead of $P$). Then there is a shortest path of form $z \cdots uv$ (a part of $P$) and a shortest path of form $z \cdots vu$ (a part of $Q$). This implies $d(u, v) = 0$, which is impossible. Part (ii) easily follows from (i). ▮

One may assume that $I$ as above is the outer face of $G$. The paths $P_1, P_2, Q_1, Q_2$ partition the space $\mathbb{R}^2 - (I \cup F)$ into four closed regions $\Omega_i$, $i = 1, 2, 3, 4$, such that $\Omega_i \cap \Omega_j = \varnothing$ for $|i - j| = 2$, and

$$\Omega_1 \cap \Omega_2 = Q_1, \qquad \Omega_2 \cap \Omega_3 = P_2, \qquad \Omega_3 \cap \Omega_4 = Q_2, \qquad \Omega_4 \cap \Omega_1 = P_1;$$

see Fig. 4. Let $\mathscr{H}_i$ be the set of holes in $\Omega_i$, and let $n_i := |\mathscr{H}_i|$. We may assume that $n_1 + n_3 \leqslant n_2 + n_4$ and $n_1 \leqslant n_3$. Since $n_1 + n_2 + n_3 + n_4 = |\mathscr{H}| - 1 \leqslant 3$, $n_1 = 0$ and $n_3 \leqslant 1$.

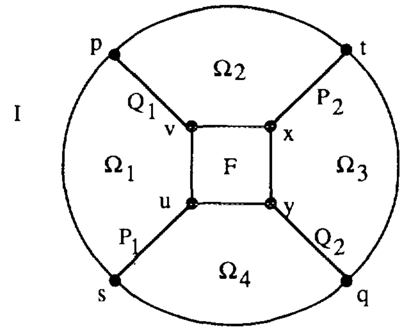


FIGURE 4

Consider the dual path $D=(K, e_1, F_1, \ldots, F_{k-1}, e_k, J)$ with $e_i = uv$ and $e_{i+1} = xy$. Since $|D \cap P| \leqslant 1$ and $|D \cap Q| \leqslant 1$, all the edges $e_1, \ldots, e_i$ lie in $\Omega_1 - (P_1 \cup Q_1)$ and all the edges $e_{i+1}, \ldots, e_k$ lie in $\Omega_3 - (P_2 \cup Q_2)$. Therefore, $K = I$, $n_3 = 1$ and $J$ is just the hole in $\Omega_3$. Next, each of $\Omega_2$ and $\Omega_4$ must contain at least one hole. For if $n_2 = 0$, say, then one of the regions in $\mathscr{R}(P)$ has no hole. But $P$ cannot lie entirely in bd$(I)$ since bd$(I)$ does not contain simultaneously $uv$ and $vx$, by Claim 1; a contradiction with Claim 2. In particular, this proves the lemma for $|\mathscr{H}| = 3$.

Now suppose that $|\mathscr{H}| = 4$. Let $x_1 x_2 \cdots x_k$ be the right boundary and $y_1 y_2 \cdots y_k$ be the left boundary of $D$; then $x_i = u$ and $y_i = v$. Choose a shortest $W$-path passing $x_1, y_1, y_2$, say $\tilde{P} = \tilde{s} \cdots x_1 y_1 y_2 \cdots \tilde{t}$, and a shortest $W$-path passing $y_1, x_1, x_2$, say $\tilde{Q} = \tilde{p} \cdots y_1 x_1 x_2 \cdots \tilde{q}$. We assert that $\tilde{s}, \tilde{t} \in V(I)$. Indeed, if $\tilde{s}, \tilde{t} \in V(I')$ for some $I' \in \mathscr{H} - \{I\}$ then at least one of $P, Q$, say $P$, meets both paths $\tilde{P}(\tilde{s}, x_1)$ and $\tilde{P}(y_2, \tilde{t})$. Hence, there is a shortest $s-t$ path passing $x_1, y_1, y_2$. Then $\tilde{p}, \tilde{q} \in V(I)$ (by Claim 3(ii) for $F_1$), whence $\tilde{s}, \tilde{t} \in V(I)$ (by the same claim); a contradiction.

So one may assume that $F$ as above is chosen to be $F_1$, and that $P = \tilde{P}$, $Q = \tilde{Q}$, $s = x_1$, $p = y_1$. Then the edges $e_1, \ldots, e_k$ of $D$ lie in $\Omega := \mathbb{R}^2 - (I \cup \Omega_2 \cup \Omega_4)$.

Let $x_{k+1}$ ($y_{k+1}$) be the vertex in bd$(J)$ opposite to $y_k$ (respectively, $x_k$). Consider the dual path $D' = (J, e_{k+1}, \ldots, e_m, I')$ with $e_{k+1} = x_{k+1} y_{k+1}$, and let $x_{k+1} \cdots x_m$ (respectively, $y_{k+1} \cdots y_m$) be the right (respectively, left) boundary of $D$; see Fig. 5.

CLAIM 4. *$d(x_i, x_j) = d(y_i, y_j)$, for all $1 \leqslant i, j \leqslant m$.*

*Proof.* Suppose to the contrary that $d(x_i, x_j) \neq d(y_i, y_j)$ for some $1 \leqslant i < j \leqslant m$. In addition, let $i, j$ be chosen so that $kj - i$ is as small as possible. From (14) (for $D$ and for $D'$) it follows that $i \leqslant k$ and $j \geqslant k+1$.

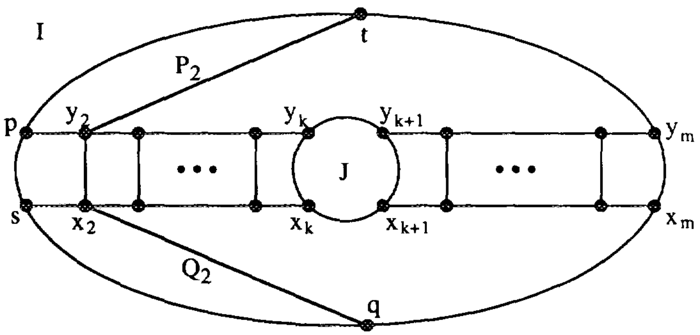


FIGURE 5

First we observe that the edges $e_2, ..., e_{j-1}$ lie in $\Omega$. For suppose that $j'$ is the minimum index such that $1 < j' < j$ and $e_{j'}$ is not in $\Omega$. Then $e_{j'}$ belongs to some of the paths $P$ and $Q$, say $Q$. From the minimality of $j'$ it follows that $Q$ passes $x_{j'}$ earlier than $y_{j'}$. Then $d(x_1, x_{j'}) < d(y_1, y_{j'})$ and $kj' - 1 < kj - i$, contrary to the choice of $i$, $j$.

By symmetry one may assume that $d(x_i, x_j) < d(y_i, y_j)$. Choose a shortest $x_i - x_j$ path $L'$, and form the path $L := y_i x_i \cdot L' \cdot y_i y_j$. Since $d(y_i, y_j) \leqslant |L| = |L'| + 2 = d(x_i, x_j) + 2$ and $d(y_i, y_j) \geqslant d(x_i, x_j) + 2$ (as $G$ is bipartite), $L$ is shortest. We may assume that $L$ lies entirely in $\Omega \cup P \cup Q$ (taking into account that $e_i, e_j$ are in $\Omega$ and that $P$ and $Q$ are shortest).

Suppose that $i = k$ and $j = k + 1$. By Claim 2 applied to $L$ and $J$, it follows that $L$ lies in the boundary of $J$. Then $|L| = \frac{1}{2}|\text{bd}(J)| + 1$ (since $y_k$ and $x_{k+1}$ are opposite on bd($J$)), whence $L$ is not shortest; a contradiction.

Now suppose that $i < k$ (the case $j > k + 1$ is symmetric). Take a shortest $r - w$ path $R = z_0 z_1 \cdots z_h$ with $\{r, w\} \in W$ passing $y_i, y_{i+1}, x_{i+1}$ (in this order); let $z_f = y_i$. If $r, w \in V(J)$, then there exists a shortest $r - w$ path $R'$ which passes $y_i, y_{i+1}, x_{i+1}$ and lies in $\Omega \cup P \cup Q$. By Claim 2 (for $R'$ and $J$), $R'$ is contained in the boundary of $J$; a contradiction with Claim 1 (for $F_i$ and $J$). Thus $r, w \in V(K)$ for some hole $K \neq J$. This means that at least one of the following is true: (a) $\tilde{R} := R(z_{f+2}, z_h)$ meets $L'$ at some vertex $z_g$; or (b) $\tilde{R}$ passes an edge $e_{j'}$ for some $i + 1 < j' < j$. In the first case, the path $R(y_i, z_g) \cdot L(z_g, y_j)$ is also shortest, whence $d(x_{i+1}, x_j) < d(y_{i+1}, y_j)$, contrary to the choice of $i$, $j$. In the second case we also obtain a contradiction with the choice of $i$, $j$. ∎

From Claim 4 it follows easily that:

(18) all vertices $x_1, ..., x_m, y_1, ..., y_m$ are distinct;

(19) the end face $I'$ of $D'$ is $I$.

By (18)–(19), the edges $e_1, ..., e_m$ form a cut $\delta X$ in $G$. We assert that $\delta X$ is reducible; this contradiction will prove the lemma. To see this, consider a $W$-path $L = \bar{s} \cdots \bar{t}$. We have to prove that

$$|L - \delta X| \geqslant d(\bar{s}, \bar{t}) - \alpha(L), \tag{20}$$

where $\alpha(L) := 1$ if $\delta X$ separates $\bar{s}$ and $\bar{t}$, and 0 otherwise. We proceed by induction on $|L|$. Inequality (20) is obvious when $|L \cap \delta X| \leqslant 1$. If $|L \cap \delta X| \geqslant 2$ then $L$ can be split as $L_1 \cdot B \cdot L_2$, where $B$ is a path $v_1 v_2 \cdots v_h$ such that $|B \cap \delta X| = 2$, $v_1 v_2 = e_i$, and $v_{h-1} v_h = e_j$ for some $1 \leqslant i, j \leqslant m$. Let for definiteness $v_1 = x_i$; then $v_h = x_j$. Choose a shortest $v_1 - v_h$ path $B'$, and put $L' := L_1 \cdot B' \cdot L_2$. By Claim 4, $|B'| \leqslant |B| - 2$. Therefore, $|L'| < |L|$ and

$$|L - \delta X| - |L' - \delta X| = |B - \delta X| - |B' - \delta X| = |B| - 2 - |B' - \delta X| \geqslant 0,$$

whence (20) follows by induction.

The proof of Lemma (2.1) is completed. ▮

Now we finish the proof of Theorem 1 (Theorem 2 follows from Lemma (2.1) and Theorem 1). A simple $s - t$ path (or circuit) $P$ in $G$ is called *elementary* if all inner vertices of $P$ are of valency 2 in $G$. If $P$ is elementary then $P$ is a part of the boundary of some hole. Since $G$ has no reducible cuts,

(21) any elementary path $P$ is shortest.

Indeed, let $P = x_0 x_1 \cdots x_k$ and $|P| > d(x_0, x_k)$. One can see that for $i := (d(x_0, x_k) + k)/2$, the cut formed by the edges $x_0 x_1$ and $x_i x_{i+1}$ is reducible.

From (21) it follows for the case $|\mathscr{H}| = 3$ that $G$ is a "theta-graph," that is, it is homeomorphic to $K_{2,3}$. Moreover, by (21), $G$ is formed by three elementary $s - t$ paths $P_i = v_i^0 v_i^1 \cdots v_i^r$, $i = 1, 2, 3$, of the same length $r$. If $r$ is an odd, say $r = 2r' + 1$, then the edges $v_i^{r'} v_i^{r'+1}$, $i = 1, 2, 3$, form a reducible cut. Thus, $r$ is an even, say $r = 2k$. Now it is a routine to check that the metric on $VG$ induced by $G$ itself is the sum of $k$ 2, 3-metrics.

This completes the proofs of Theorems 1 and 2. ▮

*Remark.* Strictly speaking, in the above proof we admited 2, 3-metrics $m$ to be induced by $\sigma$ such that $\sigma(VG) \neq VK_{2,3}$. In fact, such an $m$ can be replaced by (at most three) cut metrics. More precisely, let the vertices of $K_{2,3}$ be labelled by $t_1, t_2, s_1, s_2, s_3$ as shown in Fig. 1. For a mapping $\sigma: VG \to VK_{2,3}$ denote by $\Pi(\sigma)$ the ordered partition $(T_1, T_2, S_1, S_2, S_3)$ of $VG$ defined by

$$T_i := \sigma^{-1}(t_i), \qquad i = 1, 2 \qquad \text{and} \qquad S_j := \sigma^{-1}(s_j), \qquad j = 1, 2, 3. \tag{22}$$

For $X \subseteq VG$, let $\rho X$ denote the cut (or zero) metric on $VG$ defined by $\rho X(x, y) := 1$ if $|\{x, y\} \cap X| = 1$, and 0 otherwise. It is easy to check that if $T_j = \emptyset$ for some $j$ then the metric $m$ induced by $\sigma$ coincides with $\rho S_1 + \rho S_2 + \rho S_3$. Similarly, if $S_j = \emptyset$ for some $j$, say $S_1 = \emptyset$, then $m = \rho X + \rho Y$ for $X := T_1 \cup S_2$ and $Y := T_1 \cup S_3$. Thus, if for $m_1, ..., m_k$ satisfying (2)–(3) some $m_i$ is induced by $\sigma: VG \to VK_{2,3}$ such that $\sigma(VG) \neq VK_{2,3}$ then $m_i$ can be replaced by cut metrics, preserving validity of (2)–(3).

In conclusion note that the proof of Theorem 1 provides a polynomial algorithm for finding a required packing of cuts and 2, 3-metrics. It should be noted only that there are simple approaches to subdivide an intermediate face $F$ of the original graph into quadrangles, preserving the distances for the original vertices, in such a way that the number of new elements is bounded by a polynomial in bd($F$) (the method used in the proof gave exponential growth for this amount).

## 3. A Strengthening of Theorem 1

In this section we derive a stronger form of Theorem 1 that establishes the existence of a family $\{m_1, ..., m_k\}$ satisfying (2)–(3) in which each 2, 3-metric corresponds, in a sense, to the topological structure of the space $\mathbb{R}^2 - \bigcup (I \in \mathscr{H})$. For the purposes of Part II we shall deal with edge-weighted graphs $G$. Let us start with terminology and notation used here.

(i) Let $l$ be a nonnegative integer-valued function on $EG$; $l(e)$ is interpreted as a *length* of an edge $e$. The *l-length* $l(P)$ of a path $P = (x_0, e_1, x_1, ..., e_k, x_k)$ is $\sum_{i=1}^{k} l(e_i)$, and the $l$-distance $\text{dist}_l(x, y)$ for $x, y \in VG$ is the minimum of $l(P)$'s among all $x - y$ paths $P$ in $G$. We say that $l$ is *bipartite* if the $l$-length of each circuit in $G$ is even.

(ii) For $X \subseteq VG$, $\langle X \rangle = \langle X \rangle_G$ denotes the subgraph of $G$ induced by $X$, while $\Phi(X)$ denotes the closed region in the plane that is the union of the graph $\langle X \rangle$ and the faces of $G$ whose boundaries are entirely contained in $\langle X \rangle$.

(iii) Let $m$ be a 2, 3-metric induced by $\sigma$, and let $\Pi(\sigma) = (T_1, T_2, S_1, S_2, S_3)$ be defined as in (22). We say that $m$ is *proper* if the holes in $G$ can be labelled by $I_1, I_2, I_3$ so that the following hold:

(23) for $i = 1, 2, 3$, the graph $\langle S_i \rangle$ is connected, and it meets the boundaries of $I_{i-1}$ and $I_{i+1}$ but not the boundary of $I_i$ (taking indices modulo 3);

(24) the space $\mathbb{R}^2 - (I_1 \cup I_2 \cup I_3 \cup \Phi(S_1) \cup \Phi(S_2) \cup \Phi(S_3))$ consists of two disjoint (connected) regions one containing $T_1$ and the other containing $T_2$;

see Fig. 6. These properties imply that $G$ has no edge with one end in $T_1$ and the other in $T_2$.

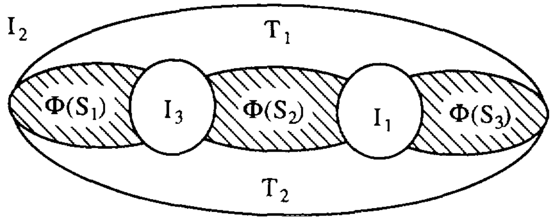


FIGURE 6

(3.1) THEOREM. *Let* $|\mathscr{H}|=3$, *and let* $l$ *be bipartite. Then there exist* $m_1, ..., m_k$ *satisfying*

$$m_1(e)+\cdots+m_k(e)\leqslant l(e) \qquad \textit{for all} \quad e\in EG; \tag{25}$$

$$m_1(s,t)+\cdots+m_k(s,t)=\operatorname{dist}_l(s,t) \qquad \textit{for all} \quad s,t\in V(I), \quad I\in\mathscr{H}, \tag{26}$$

*where each* $m_i$ *is a cut metric or a proper* 2, 3*-metric.*

*Proof.* Let $G'$ be the graph arising from $G$ by subdivision of each edge $e\in EG$ into $l(e)$ edges in series; if $l(e)=0$ this means contraction of $e$. Then $G'$ is bipartite. Let $\mathscr{H}'$ be the set of holes in $G'$ corresponding to $\mathscr{H}$. By Theorem 1 there exist $m'_1, ..., m'_k$ satisfying (2)–(3) for $G'$ and $\mathscr{H}'$, where each $m'_i$ is a cut metric or a 2, 3-metric on $VG'$. Each $m'_i$ generates in a natural way a metric $m_i$ on $VG$ such that $m_i$ is induced by $(\Gamma_i, \sigma_i)$ for $\Gamma_i\in\{K_2, K_{2,3}\}$, and (25)–(26) hold for $m_1, ..., m_k$. By the arguments given in the Remark in Section 2, one may assume that $\sigma_i(VG)=V\Gamma_i$, that is, $m_i$ is a cut metric or a 2, 3-metric.

Next, we say that a triple $\tau=(G,\mathscr{H},l)$ with $|\mathscr{H}|\leqslant 3$ and a bipartite $l\neq 0$ is *elementary* if there is no bipartite function $l'\neq 0, l$ on $EG$ such that $l'\leqslant l$ and:

$$\operatorname{dist}_{l'}(s,t)+\operatorname{dist}_{l''}(s,t)=\operatorname{dist}_l(s,t) \qquad \text{for any} \quad \{s,t\}\in W(\mathscr{H}), \tag{27}$$

where $l'':=l-l'$ and $W(\mathscr{H})$ is defined as in Section 2. Clearly if $\tau$ is elementary and $m_1, ..., m_k$ ($m_i$ is a cut metric or a 2, 3-metric) satisfy (25)–(26) for this $\tau$, then $k=1$ and, moreover, $m_1(e)=l(e)$ for all $e\in EG$ (taking into account that the restriction of a cut metric or a 2, 3-metric to $EG$ is a bipartite function). We say that $m_1$ *is associated with* $\tau$.

It suffices to show that if $\tau$ is elementary and $m^*$ is associated with $\tau$ then $m^*$ is either a cut metric or a proper 2, 3-metric. Supposing that this is false, consider a counterexample $\tau=(G,\mathscr{H},l)$ for which the value

$$\omega:=\omega(l):=\sum_{e\in EG} l(e)+|\{e\in EG\,|\,l(e)=0\}|$$

is as small as possible. Observe that:

(i) $l(e)=1$ or 2 for any $e \in EG$ (since a 2, 3-metric takes values only 0, 1 or 2, and since contraction of $e$ with $l(e)=0$ obviously yields a counterexample with smaller $\omega$);

(ii) $|\mathscr{H}|=3$ (for if $|\mathscr{H}| \leqslant 2$, (25)–(26) hold with all $m_i$'s that are cut metrics, by Schrijver's theorem);

(iii) $G$ is 2-connected (otherwise $\tau$ would be not elementary subject to (i)).

Let $\tau'=(G', \mathscr{H}', l')$ be obtained from $\tau$ by replacing each edge $e \in EG$ with $l(e)=2$ by two edges in series, each of length 1. It is easy to see that $\tau'$ is elementary as well, and a metric $m'$ on $VG'$ associated with $\tau'$ is a 2, 3-metric. Let $m'$ be induced by $\sigma'$, and $\Pi(\sigma')=(T'_1, T'_2, S'_1, S'_2, S'_3)$. Since $m'(e)=l'(e)=1$ for all $e \in EG'$, each component of $\langle T'_i \rangle$ or $\langle S'_j \rangle$ consists of a unique vertex. Consider a hole $I' \in \mathscr{H}'$. Its boundary intersects each set of $\Pi(\sigma')$ in at most one vertex (for if $s$ and $t$ are two common vertices for $\mathrm{bd}(I')$ and $B' \in \Pi(\sigma')$ then $m'(s,t)=0$ while $\mathrm{dist}_{l'}(s,t) \neq 0$). Clearly each edge in $G'$ connects a vertex in $T_i$ and a vertex in $S_j$. Thus $|\mathrm{bd}(I')|=2$ or 4 (since otherwise $T_1$ or $T_2$ would have more than two vertices in common with $\mathrm{bd}(I')$).

Consider the restriction $m$ of $m'$ to $VG$. Then $m$ is a 2, 3-metric associated with $\tau$; let $m$ be induced by $\sigma$, and $\Pi(\sigma)=(T_1, T_2, S_1, S_2, S_3)$. The above properties for $m'$ imply: for $I \in \mathscr{H}$,

(28) $|\mathrm{bd}(I)| \leqslant 4$; $|\mathrm{bd}(I) \cap B| \leqslant 1$ for each $B \in \Pi(\sigma)$; $|\mathrm{bd}(I) \cap (S_1 \cup S_2 \cup S_3)| \leqslant 2$; and if $|\mathrm{bd}(I)|=4$ then, up to cyclically shifting $\mathrm{bd}(I)=v_0v_1 \cdots v_3v_0$, one has: $v_0 \in T_1$, $v_1 \in S_i$, $v_2 \in T_2$, $v_3 \in S_{i'}$ for some $i \neq i'$.

Next, for $X, Y \subset VG$ let $[X, Y]$ denote the set of pairs $\{u, v\}$ with one element in $X$ and the other in $Y$. Suppose that there are two sets among $S_1, S_2, S_3$, say $S_2, S_3$, such that the boundary of none of $I \in \mathscr{H}$ meets both $S_2$ and $S_3$; in other words, $[S_2, S_3] \cap W=\varnothing$. Form the sets $X:=S_1 \cup T_1$ and $Y:=S_1 \cup T_2$, see Fig. 7(a). A straightforward check-up shows that

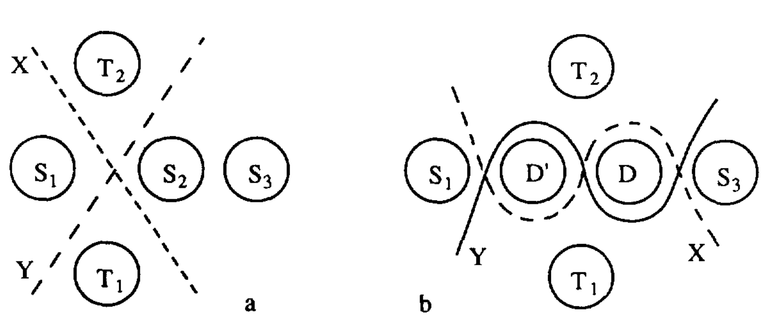


FIGURE 7

$m(x, y) \geqslant \rho X(x, y) + \rho X(x, y)$ holds for any $x, y \in VG$, and it turns into equality unless $\{x, y\} \in [S_2, S_3]$. This immediately implies that $\tau$ is not elementary.

Thus, in view of (28), each $S_i$ meets the boundaries of exactly two holes, and there is a labelling $I_1, I_2, I_3$ of the holes of $G$ so that $S_i$ meets $\text{bd}(I_j)$ for exactly those $i, j$ as described in (23). Let $s_i^j$ denote the vertex in $S_i \cap \text{bd}(I_j) \neq \varnothing$.

Suppose that for some $S_i$, say $S_2$, none of the components of $\langle S_2 \rangle$ connects the boundaries of $I_1$ and $I_3$, that is, $s_2^1 \neq s_2^3$. Form $X := S_1 \cup T_1 \cup D$ and $Y := S_3 \cup T_1 \cup D'$, where $D := \{s_2^3\}$ and $D' := S_2 - D$, see Fig. 7(b). One can check that

$$\begin{aligned} \varrho X(x, y) + \varrho Y(x, y) &= m(x, y) + 2 \qquad &&\text{if } \{x, y\} \in [D, D'], \\ &= m(x, y) - 2 &&\text{if } \{x, y\} \in [D, S_1] \cup [D', S_3], \\ &= m(x, y) &&\text{otherwise.} \end{aligned}$$

Now the fact that each of the sets $[D, D'] \cap EG$, $[S_1, D] \cap W$ and $[S_3, D'] \cap W$ is empty implies that $m$ can be replaced by $\rho X$, $\rho Y$, whence $\tau$ is not elementary.

Thus, $s_i^{i-1} = s_i^{i+1} =: s_i$ for $i = 1, 2, 3$. Obviously the space $\mathbb{R} - (I_1 \cup I_2 \cup I_3 \cup \{s_1, s_2, s_3\})$ consists of two regions $\Phi^1$ and $\Phi^2$. Let $T^i$ be the set of vertices of $G$ contained in $\Phi^i$. Then the 2, 3-metric $m^*$ corresponding to the partition $(T^1, T^2, \{s_1\}, \{s_2\}, \{s_3\})$ is proper. A routine check-up using (28) shows that $m^*(s, t) = \text{dist}_l(s, t)$ for all $\{s, t\} \in W$. ∎

## 4. Case of Four or More Holes

We say that a set $\mathscr{G}$ of connected graphs is *n-complete* if, for any $(G, \mathscr{H})$ with $G$ bipartite and $|\mathscr{H}| \leqslant n$, there exist $m_1, ..., m_k$ satisfying (2)–(3) such that each $m_i$ is a metric induced by some $\Gamma \in \mathscr{G}$ with $|E\Gamma| \leqslant |EG|$. For example, $\{K_2, K_{2,3}\}$ is 3-complete, by Theorem 1. We prove the following statement.

(4.1) (i) *Each 4-complete set contains infinitely many bipartite planar graphs $\Gamma$ with $|\mathscr{F}_\Gamma| = 4$.*

(ii) *Each 5-complete set contains a bipartite planar graphs $\Gamma$ with $|\mathscr{F}_\Gamma| > 5$.*

*Proof.* Let us say that a pair $(G, \mathscr{H})$ is *principal* if $G$ is a (connected) bipartite planar graph without multiple edges, $\mathscr{H} \subseteq \mathscr{F}_G$, and the following hold:

(29) for any distinct $x, y \in VG$, $x \neq y$, there exist $I \in \mathscr{H}$ and $s, t \in V(I)$ such that $\mathrm{dist}^{G'}(s', t') < \mathrm{dist}^{G}(s, t)$, where $G'$ is obtained from $G$ by identifying $x$ and $y$, and $s', t'$ are the images of $s, t$ in $G'$;

(30) for any $l: EG \to \mathbb{R}_+$ such that $l(e) \leqslant 1$ for each $e \in EG$ the equalities

$$\mathrm{dist}_l(s, t) + \mathrm{dist}_{1-l}(s, t) = \mathrm{dist}^G(s, t) \qquad \text{for all} \quad s, t \in V(I), \qquad I \in \mathscr{H},$$

imply that $l$ is constant.

Observe that if $(G, \mathscr{H})$ is principal then $G$ is contained in any $|\mathscr{H}|$-complete set $\mathscr{G}$. Indeed, let $m_1, ..., m_k$ satisfy (2)–(3) for given $G, \mathscr{H}$, where each $m_i$ is induced by $\sigma_i: VG \to V\Gamma_i$ for $\Gamma_i \in \mathscr{G}$ with $|E\Gamma_i| \leqslant |EG|$. Then $k = 1$ (by (30)) and $\sigma_1(x) \neq \sigma_1(y)$ for distinct $x, y \in VG$ (by (29)). Moreover, for each $xy \in EG$, the vertices $x' := \sigma_1(x)$ and $y' := \sigma_1(y)$ are connected in $\Gamma_1$ by an edge (as $0 < \mathrm{dist}^{\Gamma_1}(x', y') \leqslant m_1(x, y) \leqslant 1$, in view of $x' \neq y'$). Thus, $G$ is a subgraph of $\Gamma_1$, whence $G = \Gamma_1$.

To prove (i), consider the graph $Q$ and the weighting $w_{p,q}$ (depending on natural $p, q$) on its edges as drawn in Fig. 8(a). Denote by $G_{p,q}$ the graph obtained from $Q$ by partitioning each $e \in EQ$ into $w_{p,q}(e)$ edges in series. We assert that if the g.c.d. of $p, q$ is 1 then the pair $(G, \mathscr{H})$ is principal, where $G := G_{p,q}$ and $\mathscr{H} = \mathscr{F}_G$. Check-up of (29) (for arbitrary $p, q$) is easy.

To see (30), we use a method similar to that developed in [7, 1] for proving the primitivity of the metrics of certain graphs. Let $l$ satisfy the equalities in (30); then any shortest path in $G$ which connects some $s, t \in V(I)$, $I \in \mathscr{H}$, must be shortest with respect to $l$. One can see that for each face $F$ with $\mathrm{bd}(F) = x_0 x_1 \cdots x_{2k-1} x_0$ $(k := p + q)$, and for $i = 0, 1, ..., 2k - 1$, the path $P_i := x_i x_{i+1} \cdots x_{k+i}$ is shortest in $G$ (taking indices modulo $2k$). Considering four shortest paths $P_i, P_{i+1}, P_{i+k}, P_{i+k+1}$ we conclude that $l(x_i x_{i+1}) = l(x_{i+k} x_{i+k+1})$. So $l(e) = l(e')$ holds for any two opposite edges $e, e'$ in $\mathrm{bd}(I)$. Now using the fact that the g.c.d. of $p, q$ is 1 one can show that, for any $e, e' \in EG$, there is a sequence $e_1, e_2, ..., e_r \in EG$ such that $e = e_1$, $e' = e_r$ and each two $e_j, e_{j+1}$ are opposite on the boundary of some face. Therefore, $l$ is constant.

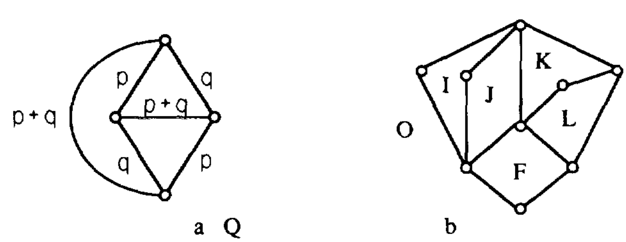


FIGURE 8

Validity of (ii) is provided by the graph $G$ with $|\mathscr{F}_G| = 6$ and $\mathscr{H} := \{O, I, J, K, L\}$, as illustrated in Fig. 8(b). Verification of (29) is straightforward. The proof of (30) uses arguments as above and it is left to the reader. (Instructions: observe that for any $F' \in \mathscr{F}_G$ with $\text{bd}(F') = x_0 x_1 \cdots x_{2k-1} x_0$, each path $x_i \cdots x_{i+k}$ is a part of a shortest path in $G$ connecting some $s, t \in I'$, $I' \in \mathscr{H}$.) ∎

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