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Paths and Metrics in a Planar Graph with Three or More Holes. II. Paths

A. V. KARZANOV*

Institute for System Analysis of Russian Academy of Science, 9, Prospect 60 Let Oktyabrya, 117312 Moscow, Russia

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Suppose that $G=(VG, EG)$ is a planar graph embedded in the euclidean plane, that I, J, K are three of its faces (*holes*), that $s_1, ..., s_r, t_1, ..., t_r$ are vertices of G such that each pair $\{s_i, t_i\}$ belongs to the boundary of some of I, J, K, and that the graph $(VG, EG \cup \{\{s_1, t_1\}, ..., \{s_r, t_r\}\})$ is eulerian. We prove that there exist edge-disjoint paths $P_1, ..., P_r$ in G such that each P_i connects s_i and t_i if the obvious necessary conditions with respect to the cuts and the so-called 2, 3-metrics are satisfied. In particular, such paths exist if the corresponding (fractional) multicommodity flow problem has a solution. This extends Okamura's theorem on paths in a planar graph with two holes. The proof uses a theorem on a packing of cuts and 2, 3-metrics obtained in Part I of the present series of two papers. We also exhibit an instance with four holes for which the multicommodity flow problem is solvable but the required paths do not exist.

1. INTRODUCTION

Throughout, we deal with an undirected planar graph G embedded in the euclidean plane $\mathbb{R}^2$. VG is the vertex set, EG is the edge set of G (multiple edges and loops are admitted), and $\mathscr{F} = \mathscr{F}_G$ is the set of faces of G. A subset $\mathscr{H} \subseteq \mathscr{F}$ of faces of G, called its *holes*, is distinguished. Let $U = \{\{s_1, t_1\}, ..., \{s_r, t_r\}\}$ be a family of pairs (possibly repeated) of vertices of G such that each $\{s_i, t_i\}$ is contained in the boundary bd(I) of some hole $I \in \mathscr{H}$.

Problem (G, U, k). Given an integer $k \geqslant 1$, find $P_1^1, ..., P_1^k, P_2^1, ..., P_2^k, ..., P_r^1, ..., P_r^k$ such that each P_i^j is a path in G connecting s_i and t_i, and each edge of G occurs at most k times in these paths.

The problem $(G, U, 1)$ is denoted by (G, U); it consists in finding edge-disjoint paths $P_1, ..., P_r$ in G, where P_i connects s_i and t_i.

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For $X \subseteq V$, let $\delta X = \delta^G X$ denote the set of edges of G with one end in X and the other in $VG - X$; a nonempty set δX is called a *cut* in G; we say that X (or δX) *separates* vertices x and y if exactly one of x, y is in X.

We prove the following theorem.

THEOREM 1. *Let* $|\mathscr{H}| = 3$, *and let*

(1) $|\delta X| + |\{i \mid X$ *separates* s_i *and* $t_i\}|$ *be even for any* $X \subset VG$.

Then (G, U) *has a solution* (*that is, required paths exist*) *if and only if*:

(2) *each* $X \subset VG$ *separates at most* $|\delta X|$ *pairs in* U;

(3) $\sum_{e \in EG} m(e) \geq \sum_{i=1}^r m(s_i, t_i)$ *for all* 2, 3-*metrics* m *on* VG.

(A 2, 3-*metric* on VG is a function m on $VG \times VG$ such that $m(x, y)$ is equal to $\mathrm{dist}^{K_{2,3}}(\sigma(x), \sigma(y))$ for some mapping σ of VG *onto* the vertex-set of the complete bipartite graph $K_{2,3}$; here $\mathrm{dist}^{G'}(u, v)$ is the distance between vertices u and v in a graph G'. We say that m *is induced by* σ, and denote m as m_σ.) Obviously, (2) is necessary for the solvability of (G, U, k) with any k. (3) is necessary as well because if P_i^j's as above give a solution of (G, U, k) then

$$\sum_{e \in EG} m(e) \geq \frac{1}{k} \sum_{i=1}^{r} \sum_{j=1}^{k} \sum (m(e) \mid e \in P_i^j) \geq \sum_{i=1}^{r} m(s_i, t_i),$$

since m satisfies the triangle inequalities (here we write $e \in P_i^j$ considering a path as an edge-set). Hence, Theorem 1 has the following corollary.

1.1. *If* $|\mathscr{H}| = 3$, (1) *holds, and* (G, U, k) *has a solution for some* k, *then* (G, U) *has a solution as well.*

Okamura [2] showed that for $|\mathscr{H}| = 2$, (G, U) has a solution whenever (1)–(2) hold (for $|\mathscr{H}| = 1$ this was shown in [3]). For $|\mathscr{H}| = 3$, (1)–(2) are, in general, not sufficient for the solvability of (G, U, k); e.g., consider $G = K_{2,3}$ and $U := \{\{s^1, s^2\}, \{s^2, s^3\}, \{s^3, s^1\}, \{t^1, t^2\}\}$, see Fig. 1a. The essence of Theorem 1 is that for $|\mathscr{H}| = 3$ adding (3) to (1)–(2) ensures the solvability of (G, U, k) with any k.

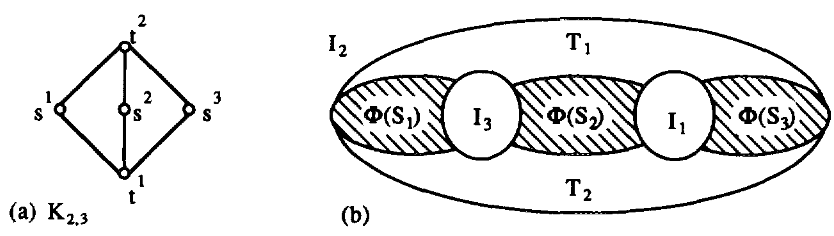


FIGURE 1

The proof of the theorem relies on the following result on packings of cuts and 2, 3-metrics obtained in Part I [1]. Let $\text{dist}_l(x, y)$ denote the distance between vertices x and y in G whose edges $e \in EG$ have *lengths* $l(e) \in \mathbb{Q}_+$ ($\mathbb{Q}_+$ is the set of nonnegative rationals). A mapping $\sigma: VG \to VK_{2,3}$ determines the ordered partition $\Pi(\sigma) = (T_1, T_2, S_1, S_2, S_3)$ of VG, where $T_i := \sigma^{-1}(t^i)$, $S_j := \sigma^{-1}(s^j)$, and the vertices of $K_{2,3}$ are labelled by t^1, t^2, s^1, s^2, s^3 as shown in Fig. 1a. Let $\langle X \rangle = \langle X \rangle_G$ denote the subgraph of G induced by $X \subseteq VG$, and let $V(F)$ denote the set of vertices in $\text{bd}(F)$, $F \in \mathscr{F}$. By a *region* we mean a connected set in the plane that is the union of some vertices, edges, and faces of G; an edge (a face) of G is identified with the corresponding curve without the endpoints (respectively, the corresponding open two-dimensional set) in the plane.

One sort of 2, 3-metrics will be important in what follows. Let m be a 2, 3-metric induced by a mapping σ. We call σ (as well as m and $\Pi(\sigma)$) *proper* if for some labelling I_1, I_2, I_3 of the holes of G, the partition $\Pi(\sigma) = (T_1, T_2, S_1, S_2, S_3)$ satisfies:

(4) for $i = 1, 2, 3$, $\langle S_i \rangle$ is connected, and $S_i \cap V(I_p) = \varnothing$ if and only if $p = i$;

(5) the space $\Omega(\sigma) := \mathbb{R}^2 - (I_1 \cup I_2 \cup I_3 \cup \Phi(S_1) \cup \Phi(S_2) \cup \Phi(S_3))$ consists of two disjoint regions, one containing T_1 and the other containing T_2; here $\Phi(S_i)$ is the union of $\langle S_i \rangle$ and the faces F of G with $\text{bd}(F) \subseteq \langle S_i \rangle$.

(See Fig. 1b.) In particular, (5) implies that no edge of G connects T_1 and T_2.

THEOREM 2 [1]. *Let G be connected, $|\mathscr{H}| = 3$, and $l: EG \to \mathbb{Q}_+$. Then there exist cuts $\delta X_1, ..., \delta X_N$ in G, proper 2, 3-metrics $m_1, ..., m_M$ on VG, and numbers $\lambda_1, ..., \lambda_N, \mu_1, ..., \mu_M \in \mathbb{Q}_+$ such that*

$$\sum (\lambda_i \mid i = 1, ..., N, e \in \delta X_i) + \sum_{j=1}^{M} \mu_j m_j(e) \leqslant l(e) \quad \textit{for all} \quad e \in EG; \tag{6}$$

$$\sum (\lambda_i \mid i = 1, ..., N, X_i \textit{ separates } s \textit{ and } t) + \sum_{j=1}^{M} \mu_j m_j(s, t) = \text{dist}_l(s, t) \quad \textit{for all} \quad s, t \in V(I), \quad I \in \mathscr{H}. \tag{7}$$

In Section 2 from Theorem 2 we obtain a criterion of the solvability of (G, U, k) for some k and then, in Section 3, using this criterion, we shall prove Theorem 1.

Note that in Part I it was shown also that for the case $|\mathscr{H}| = 4$, (6)–(7) can be satisfied by taking as m_i's 2, 3-metrics or metrics induced by mappings into planar graphs with four faces. Using arguments from Section 2, for this case one can derive from that result a combinatorial criterion of the solvability of (G, U, k) for some k. However, (1.1) does not remain true for $|\mathscr{H}| = 4$, as we explain in Section 4.

2. Multicommodity Flows and Metrics

Standard linear programming duality arguments enable us to obtain from Theorem 2 a weaker, fractional, version of Theorem 1. Let G, $\mathscr{H}$, U be as in the hypotheses of Theorem 1.

Denote by $\mathscr{P}_i = \mathscr{P}(G, s_i, t_i)$ the set of simple paths from s_i to t_i (or $s_i - t_i$ *paths*) in G. Let $\mathscr{P} = \mathscr{P}(G, U) := \bigcup (\mathscr{P}_i \mid i = 1, ..., r)$.

Problem (c, g). Given a function $c: EG \to \mathbb{Q}_+$ (of *capacities* of edges) and numbers $g_1, ..., g_r \in \mathbb{Q}_+$ (*demands*), find a function $f: \mathscr{P} \to \mathbb{Q}_+$ satisfying:

$$f^e := \sum (f(P) \mid e \in P \in \mathscr{P}) \leqslant c(e) \qquad \text{for all} \quad e \in EG; \tag{8}$$

$$\sum (f(P) \mid P \in \mathscr{P}_i) = g_i \qquad \text{for} \quad i = 1, ..., r. \tag{9}$$

Such an f is called a (c, g)-admissible *multicommodity flow*. For $c \equiv 1$ and $g \equiv 1$, (c, g) with the additional requirement on f to be integer-valued turns into the above problem (G, U).

By Farkas lemma, (8)–(9) is solvable if and only if

$$cl := \sum_{e \in EG} c(e)\, l(e) \geqslant \sum_{i=1}^{r} g_i b_i \tag{10}$$

holds for any $l \in \mathbb{Q}_+^{EG}$ and $b_1, ..., b_r \in \mathbb{Q}$ satisfying

$$l(P) := \sum (l(e) \mid e \in P) \geqslant b_i, \qquad P \in \mathscr{P}_i, \quad i = 1, ..., r. \tag{11}$$

Since (11) is equivalent to $b_i \leqslant \mathrm{dist}_l(s_i, t_i)$ $(i = 1, ..., r)$, (c, g) is solvable if and only if

$$cl \geqslant \sum_{i=1}^{r} g_i \,\mathrm{dist}_l(s_i, t_i) \tag{12}$$

holds for any $l \in \mathbb{Q}_+^{EG}$. For a fixed l choose X_i's, λ_i's, m_j's, μ_j's as in Theorem 2. Define

$$c(X_j) := \sum (c(e) \mid e \in \delta X_j); \qquad g(X_j) := \sum (g_i \rho X_j(s_i, t_i) \mid i = 1, ..., r);$$

$$c(m_q) := \sum (c(e)\, m_q(e) \mid e \in EG); \qquad g(m_q) := \sum (g_i m_q(s_i, t_i) \mid i = 1, ..., r),$$

where for $X \subseteq VG$, $\rho X(x, y)$ denotes the function on $VG \times VG$ taking the value one if X separates x and y, and zero otherwise. Then (6) implies

$$cl \geq \lambda_1 c(X_1) + \cdots + \lambda_N c(X_N) + \mu_1 c(m_1) + \cdots + \mu_M c(m_M),$$

while (7) implies

$$\sum_{i=1}^{r} g_i \operatorname{dist}_l(s_i, t_i) = \sum_{j=1}^{N} \lambda_j g(X_j) + \sum_{q=1}^{M} \mu_q g(m_q),$$

whence (by (12)) we decude the following statement:

2.1. *(c, g) is solvable if and only if the following hold*:

$$c(X) \geq g(X) \qquad \textit{for any} \quad X \subset VG; \tag{13}$$

$$c(m) \geq g(m) \qquad \textit{for any proper } 2, 3\textit{-metric } m \textit{ on } VG. \tag{14}$$

(The "only if" part follows from the facts that if $c(X) < g(X)$ for some $X \subset VG$ then, obviously, (12) is violated for $l := \rho X|_{EG}$, and similarly, if $c(m) < g(m)$ for some 2, 3-metric m on VG then (12) is violated for $l := m|_{EG}$.)

3. Proof of Theorem 1

Let G, $\mathscr{H}$, U be as the hypotheses of Theorem 1, and let (1)–(3) hold. Put $c(e) := 1$ for $e \in EG$ and $g_i := 1$ for $i = 1, ..., r$. Then (2)–(3) is equivalent to (13)–(14). Therefore (by (2.1)), the problem (c, g) has a solution. One must prove that (c, g) has an integral solution (the "only if" part in Theorem 1 was explained in the Introduction).

Without loss of generality one may assume that: G is connected; the outer (unbounded) face of G is a hole; all s_i's and t_j's are distinct and have valency 1 (since for $i = 1, ..., r$ one can add new vertices s_i', t_i' and edges $\{s_i', s_i\}$, $\{t_i', t_i\}$ and consider the pair $\{s_i', t_i'\}$ instead of $\{s_i, t_i\}$). Let $T := \{s_1, ..., s_r, t_1, ..., t_r\}$.

Next, one may assume that each vertex in $VG - T$ is of valuency 2 or 4. For if $x \in VG - T$ has valency $h > 4$, one can transform G at x as shown in Fig. 2 (it is easy to see that such a transformation yields an equivalent problem).

Figure 2

We proceed by induction on $|EG|$, assuming $|\mathscr{H}| \leqslant 3$. If G has a loop or a vertex of valency 2 in $VG - T$, the result obviously follows by induction; while if $|\mathscr{H}| \leqslant 2$ or $T \cap V(I) = \varnothing$ for some $I \in \mathscr{H}$, the result follows from Okamura's theorem.

The proof falls into two parts. We first prove the existence of a *half-integral* solution for (c, g); using it, we then show that it has an integral solution as well.

Some conventions. By a *circuit* we mean an arbitrary x–x path. When it leads to no confusion we identify a path (circuit) in G and its image in the plane. The boundary $\mathrm{bd}(F)$ of a face F will be often considered as a (possibly not simple) circuit oriented clockwise from a point in F. For $x \in VG$, $E(x)$ denotes the clockwise-ordered sequence (considered up to a shifting cyclically) of the edges incident to x.

Consider a vertex $x \in VG - T$ and two consecutive edges $e, e' \in E(x)$. The triple $\tau = (e, x, e')$ is called a *fork*. Denote by G_τ the (planar) graph obtained from G by adding a new edge (or a loop) e_τ connecting the ends of the edges e and e' different from x. Define the function ω_τ on EG_τ by

$$\begin{aligned} \omega_\tau(u) &:= 1 && \text{for} \quad u = e, e', \\ &:= -1 && \text{for} \quad u = e_\tau, \\ &:= 0 && \text{otherwise.} \end{aligned}$$

For $0 \leqslant \varepsilon \leqslant 1$, let $c_{\tau,\varepsilon}$ denote the function on EG_τ taking the value $1 - \varepsilon$ on e and e', ε on e_τ, and 1 on the edges in $EG - \{e, e'\}$. We say that ε is *feasible* if the problem $(c_{\tau,\varepsilon}, g)$ has a solution, or, in other words, if (13)–(14) hold for G_τ and $c_{\tau,\varepsilon}$. E.g., $\varepsilon = 0$ is feasible. The maximum feasible $\varepsilon \leqslant 1$ is denoted by $\alpha(\tau)$.

Our main aim is to prove the existence of a fork τ such that $\alpha(\tau) = 1$. Then the proof of Theorem 1 is completed as follows. Let G' be the graph arising from G by deleting e, e' and adding e_τ. Then the solvability of $(c_{\tau,1}, g)$ means that (2)–(3) hold for G' and U. Since $|EG'| = |EG| - 1$, and (G', U) satisfy (1), the result for (G, U) easily follows by induction.

Thus, one may assume that $\alpha(\tau) < 1$ for all forks τ in G. Denote by $\mathscr{M}$ the set of all proper 2, 3-metrics on VG. For $0 \leqslant \varepsilon \leqslant 1$ and $X \subset VG$ (respectively, $m \in \mathscr{M}$), put $\Delta_{\tau,\varepsilon}(X) := c_{\tau,\varepsilon}(X) - g(X)$ (respectively, $\Delta_{\tau,\varepsilon}(m) := c_{\tau,\varepsilon}(m) - g(m)$). Then

$$\Delta_{\tau,\varepsilon}(X) = c(X) - g(X) - \varepsilon\omega_\tau(X), \tag{15}$$

$$\Delta_{\tau,\varepsilon}(m) = c(m) - g(m) - \varepsilon\omega_\tau(m), \tag{16}$$

where $\omega_\tau(X)$ stands for $\rho X(e) + \rho X(e') - \rho X(e_\tau)$, and $\omega_\tau(m)$ stands for $m(e) + m(e') - m(e_\tau)$. Observe that: (i) $\Delta_{\tau,0}(X) \geqslant 0$ and $\Delta_{\tau,0}(m) \geqslant 0$ (by

(13)–(14)); (ii) $\omega_\tau(X)\geqslant 0$ and $\omega_\tau(m)\geqslant 0$ (since ρX and m are metrics); and (iii) if ε' is such that $\alpha(\tau)<\varepsilon'\leqslant 1$ then there is $X\subset VG$ such that $\Delta_{\tau,\varepsilon'}(X)<0$ or there is $m\in\mathcal{M}$ such that $\Delta_{\tau,\varepsilon'}(m)<0$ (by the definition of $\alpha(\tau)$). This implies that $\alpha(\tau)$ satisfies

$$\alpha(\tau)=\min\{\min\{(c(X)-g(X))/\omega_\tau(X)\mid X\subset VG,\ \omega_\tau(X)>0\}$$
$$\min\{(c(m)-g(m))/\omega_\tau(m)\mid m\in\mathcal{M},\ \omega_\tau(m)>0\}\}. \quad (17)$$

We say that $X\subset VG$ (or $m\in\mathcal{M}$) is *crucial* for τ if it achieves the minimum in (17).

3.1. *If $\alpha(\tau)>0$ then no set $X\subset VG$ is crucial for τ.*

Proof. Let $X\subset VG$ and $b:=\omega_\tau(X)>0$. Since $\alpha(\tau)>0$, $a:=c(X)-g(X)>0$. Hence $a\geqslant 2$ (by (1)). On the other hand, $b\leqslant 2$. Thus, $a/b\geqslant 1>\alpha(\tau)$. ∎

Remark. A simple fact (implied, e.g., by (3.5) below) is that if $\alpha(\tau)=0$ holds for all forks τ then (c, g) has a unique solution f and, moreover, f is integral. Thus (cf., e.g., [4]), (3.1) enables to derive Okamura's theorem directly from its fractional version: if $|\mathcal{H}|\leqslant 2$ and (13) holds then (c, g) has a solution.

3.2. *Let $\alpha(\tau)>0$, and let $m\in\mathcal{M}$ be crucial for τ. Then $\alpha(\tau)=\frac{1}{2}$, $c(m)-g(m)=2$, and $\omega_\tau(m)=4$.*

Proof. Put $b:=\omega_\tau(m)$ and $a:=c(m)-g(m)$. Then $b>0$ and $\alpha(\tau)=a/b$. From (1) it easily follows that a is even. Next, since $m(y,z)\leqslant 2$ for any $y, z\in VG$ and every ciruit in $K_{2,3}$ is even, b is even and $b\leqslant 4$. In view of $0<\alpha(\tau)<1$, only one case is possible, namely, $a=2$ and $b=4$.

Thus, $\alpha(\tau)\in\{0,\frac{1}{2}\}$ for any fork τ. Let us fix a multicommodity flow $f\colon\mathcal{P}(G,U)\to\mathbb{Q}_+$ that is a solution of (c, g). It will be convenient to think of f as consisting of three "flows" f_I, f_J, and f_K, where $\mathcal{H}=\{I,J,K\}$, and f_F is the restriction of f to the set of paths in $\mathcal{P}(G,U)$ with both ends in $V(F)$, $F\in\mathcal{H}$. Denote by $\mathcal{L}=\mathcal{L}(f)$ the set of paths $P\in\mathcal{P}(G,U)$ with $f(P)>0$ (the *support* of f). Similarly, $\mathcal{L}_F=\mathcal{L}_F(f)$ denotes the support of f_F; so $\{\mathcal{L}_I,\mathcal{L}_J,\mathcal{L}_K\}$ is a partition of $\mathcal{L}$.

A path $P\in\mathcal{L}_F(F\in\mathcal{H})$ splits the space $\mathbb{R}^2-F$ into the pair $\mathcal{R}(P)$ of closed regions whose intersection is P and union is $\mathbb{R}^2-F$. We say that f is *regular* if any $P\in\mathcal{L}_F$ and $P'\in\mathcal{L}_{F'}$ for $F\neq F'$ do not cross, that is, P' is contained entirely in some Ω_i, where $\mathcal{R}(P)=\{\Omega_1,\Omega_2\}$. The following property is shown by use of standard uncrossing techniques.

3.3. *If f is $1/k$-integral then (c, g) has a $1/k$-integral solution f' that is regular.*

In what follows we assume that f is regular. Consider a hole, say I. Denote by Ψ_1 the region that is the union of $Q_I := \mathrm{bd}(I) \cup \bigcup (P \in \mathscr{L}_I)$ and those components of $\mathbb{R}^2 - Q_I$ which contain no hole; call Ψ_I *the region of the flow* f_I. Each component Z of $\mathbb{R}^2 - (I \cup \Psi_I)$ contains at least one hole among J, K; moreover, the fact that every path path in $\mathscr{L}_I$ is simple implies that the boundary of Z is formed by a simple circuit C. If $J \subseteq Z$, say, we denote C by $C_I(J)$. If $C_I(J) = C_I(K)$ then $C_I(J)$ is denoted by C_I. If $C_I(J) \neq C_I(K)$, we say that f_I is *separating*. In view of the regularity of f,

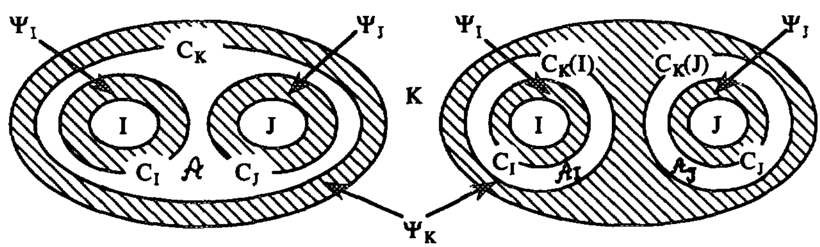
(18) (i) at least two of f_I, f_J, f_K are non-separating;

(ii) for $F, F' \in \mathscr{H}$ $(F \neq F')$, the sets $\Psi_F - C_F(F')$ and $\Psi_{F'} - C_{F'}(F)$ are disjoint.

(See Fig. 3.) For $F \in \mathscr{H}$ and $e \in EG$ put $f_F^e := \sum (f(P) \mid e \in P \in \mathscr{L}_F)$. We shall prove the following lemma.

3.4. LEMMA. *Let f_I be non-separating. Then for any edge e in the circuit C_I, at least one of the values f_I^e and $f_J^e + f_K^e$ is zero.*

In the assumption that Lemma 3.4 is valid the existence of a half-integral solution for (c, g) is proved as follows. Let G_1 (G_2) be the subgraph of G contained in Ψ_I (respectively, in $\Psi_J \cup \Psi_K$), and let U_1 (U_2) be the set of pairs $\{s, t\} \in U$ with $s, t \in V(I)$ (respectively, $s, t \in V(J) \cup V(K)$). For $i = 1, 2$, define the capacity $c_i(e)$ of an edge $e \in EG_i$ to be two if $f_i^e > 0$, and zero otherwise; here $f_1^e := f_I^e$ and $f_2^e := f_J^e + f_K^e$. Put $g_i(\{s, t\}) := 2$ for $\{s, t\} \in U_i$. Since $f_J^e + f_K^e \leq \frac{1}{2} c_2(e)$ for each $e \in EG_2$, the flows $2f_J$ and $2f_K$ determine a solution for (c_2, g_2). Furthermore, the functions c_2 and g_2 take even values and each pair $\{s, t\} \in U_2$ is contained in the boundary of some of two faces of G_2. Thus, by Okamura's theorem (c_2, g_2) has an integral solution ϕ_2. Similarly, (c_1, g_1) has an integral solution ϕ_1.



(a) f_I, f_J, f_K are non-separating (a) f_K is separating

FIGURE 3

We know that $G_1 \cap G_2 \subseteq C_l$ (by (18)(ii)), and $c_1(e)+c_2(e) \leqslant 2 = 2c(e)$ for any edge e in C_l (by (3.4)). Thus, $\frac{1}{2}\phi_1$ and $\frac{1}{2}\phi_2$ determine a half-integral solution of (c, g).

In order to prove (3.4) we need the auxiliary statements (3.5)–(3.8); they will be also used to show the existence of an integral solution for (c, g).

For $e \in EG$ ($e, e' \in EG$), denote by $\mathscr{D}(e) = \mathscr{D}_f(e)$ (respectively, $\mathscr{D}(e, e') = \mathscr{D}_f(e, e')$) the set of paths in $\mathscr{L}$ containing e (respectively, e and e'). Put $f^e := \sum (f(P) \mid \in \mathscr{D}(e))$ (cf. (8)) and $f^{e,e'} := \sum (f(P) \mid P \in \mathscr{D}(e, e'))$. For a fork $\tau = (e, x, e')$ introduce the value $\beta(\tau)$ which, as we shall see later, gives a lower bound for $\alpha(\tau)$:

$$\beta(\tau) := 1 - \tfrac{1}{2} f^e - \tfrac{1}{2} f^{e'} + f^{e,e'} \qquad (= 1 - \tfrac{1}{2}(f^{e,u} + f^{e,u'} + f^{e',u} + f^{e',u'})),$$

where $E(x) = (e, e', u, u')$. By symmetry,

$$\beta(e, x, e') = \beta(u, x, u') \qquad \text{for} \quad E(x) = (e, e', u, u'). \tag{19}$$

3.5. $\beta(\tau) \leqslant \alpha(\tau)$.

Proof. Let for definiteness $f^e \geqslant f^{e'}$. Define the capacity function c' on EG_τ as: $c'(e) := f^e - f^{e,e'}$; $c'(e') := f^{e'} - f^{e,e'}$; $c'(e_\tau) := 1 + f^{e,e'} - f^e$; and $c(w) := c(w)$ for the other edges w. It is easy to see that (c', g) has a solution. Now put $c'' := c_{\tau, \beta(\tau)}$ and $\varepsilon := (f^e - f^{e'})/2$. A straightforward checkup shows that $c''(w) - c'(w)$ is equal to ε for $w = e', e_\tau$; $-\varepsilon$ for $w = e$; and zero for the other $w \in EG_\tau$. Since $\varepsilon > 0$, the solvability for (c', g) implies that for (c'', g). Hence $\alpha(\tau) \geqslant \beta(\tau)$. ∎

Thus, $\beta(\tau) \leqslant \frac{1}{2}$ for all forks τ in G. In particular, this implies that G has no multiple edges. For suppose that two edges e, u' have the same ends x, y. Without loss of generality one may assume that G is embedded in the plane so that u', e are consecutive in $E(x)$, that each path in $\mathscr{D}(e)$ passes e', and each path in $\mathscr{D}(u')$ passes u, where $E(x) = (e, e', u, u')$. Then $f^{e,u'} = f^{e,u} = f^{e',u'} = 0$, whence $1 \geqslant 2\beta(e, x, e') = 2 - f^{e',u}$. Hence, $\beta(e', x, u) \geqslant f^{e',u} = 1$, a contradiction.

Note also that $\beta(e, x, e') = 0$ would imply $f^{e,e'} = 0$. Therefore, if $\beta(\tau) = 0$ for all forks τ in G then any two paths in $\mathscr{L}$ having a common edge must coincide, whence f is integer-valued.

In what follows an edge in G with ends x and y (a path $P = (x_0, e_1, x_1, ..., e_p, x_p)$) may be denoted by xy (respectively, by $x_0 x_1 \cdots x_p$).

Let us fix a fork $\tau = (e, x, e')$; let $E(x) = (e, e', u, u')$. Statement (3.6) exhibits a situation (which will take place often later on) when $\beta(\tau) = \frac{1}{2}$) occurs, while (3.7) and (3.8) describe important properties of τ with $\beta(\tau) = \frac{1}{2}$.

3.6. *Let $\mathscr{D}(e, u) = \varnothing$. Then $\beta(\tau) = \frac{1}{2}$, $f^{e,e'} = f^{e,u'}$, and the edges e' and u' are saturated by f.*

Proof. We have $2\beta(\tau) = 2 - f^{e,u'} - f^{e',u} - f^{e',u'}$ (since $f^{e,u} = 0$) and $1 = c(e') \geqslant f^{e'} = f^{e',e} + f^{e',u} + f^{e',u'}$. This implies $1 \geqslant 2\beta(\tau) \geqslant 1 + f^{e,e'} - f^{e,u'}$, whence $f^{e,e'} \leqslant f^{e,u'}$. Similarly, considering the fork (u', x, e) we obtain $f^{e,u'} \leqslant f^{e,e'}$. Thus, equality should hold throughout, and the result follows ($f^{u'} = 1$ is shown similarly to $f^{e'} = 1$). ▮

Consider a metric $m = m_\sigma \in \mathcal{M}$ crucial for τ. Let $\Pi(\sigma) = (T_1, T_2, S_1, S_2, S_3)$, and let $e = xy$ and $e' = xz$.

3.7. *Let* $\beta(\tau) = \frac{1}{2}$. *Then*

(i) $x \in S_i$ *and* $y, z \in S_{i'}$ *for some* $i \neq i'$;

(ii) *each path in* $\mathcal{L} - \mathcal{D}(e, e')$ *is shortest with respect to* m;

(iii) *each* $h \in EG - \{e, e'\}$ *with* $m(h) > 0$ *is saturated by* f; *that is,* $f^h = 1$.

Proof. By (3.2), $\omega_\tau(m) = m(e) + m(e') - m(y, z) = 4$, whence $m(e) = m(e') = 2$ and $m(y, z) = 0$. This implies that either (a) $x \in T_j$, $y, z \in T_{3-j}$ for some $j \in \{1, 2\}$, or (b) $x \in S_i$, $y, z \in S_{i'}$ for some $i \neq i'$. By (5), (a) is impossible; thus (i) is true. Next, for an $s-t$ path P put $\mu(P) := \sum_{h \in P} m(h) - m(s, t)$. Since m is a metric, $\mu(P) \geqslant 0$ for all $P \in \mathcal{L}$. Furthermore, $m(e) = m(e') = 2$ and $m(y, z) = 0$ imply that $\mu(P) \geqslant 4$ for all $P \in \mathcal{D}(e, e')$. We have (by (3.2))

$$
\begin{aligned}
2 = c(m) - g(m) &= \sum_{h \in EG} m(h) - g(m) \\
&= \sum_{h \in EG} m(h)(1 - f^h) + \sum_{h \in EG} m(h) f^h - g(m) \\
&\geqslant m(e)(1 - f^e) + m(e')(1 - f^{e'}) + \sum_{h \in EG} m(h) f^h - g(m) \\
&= 4 - 2f^e - 2f^{e'} + \sum_{P \in \mathcal{L}} \mu(P) f(P) \\
&\geqslant 4 - 2f^e - 2f^{e'} + 4 \cdot \sum_{P \in \mathcal{D}(e, e')} f(P) \\
&= 4 - 2f^e - 2f^{e'} + 4f^{e,e'} = 4\beta(\tau) = 2.
\end{aligned}
\tag{20}
$$

Hence, all the inequalities in (20) hold with equality, whence (ii) and (iii) follow. ▮

For $\Pi(\sigma) = (T_1, T_2, S_1, S_2, S_3)$ as above, let $\Phi(T_i)$ denote the component of $\Omega(\sigma)$ containing T_i, $i = 1, 2$ (where $\Omega(\sigma)$ is defined in (5)). For a path $P = x_0 x_1 \cdots x_p$ and $0 \leqslant j \leqslant j' \leqslant p$, the part of P from x_j to $x_{j'}$ is denoted by $P(x_j, x_{j'})$. For $x' \in VG$ let $\mathcal{P}(x')$ denote the set of paths in $\mathcal{L}$ passing x'.

3.8. *Let $\beta(\tau)=\frac{1}{2}$. Let i and i' be as in* (3.7). *Suppose that I is the hole such that $S_i \cap V(I) \neq \varnothing$ and $S_{i'} \cap V(I) \neq \varnothing$. Then each path in $\mathscr{P}(x)$ belongs to $\mathscr{L}_I$.*

Proof. Consider a path in $\mathscr{L}$ containing exactly one of the edges e, e', say $P=s \cdots xy \cdots t$. Then P is shortest for m (by 3.7(ii)), $m(x, y)=2$ and $m(s, t) \leqslant 2$, whence all vertices of $P(s, x)$ are in S_i and those of $P(y, t)$ are in $S_{i'}$. In particular, $s, t \in T \cap V(I)$ (by (4)), that is, $P \in \mathscr{L}_I$. Note also that the facts that $P(s, x)$ lies in $\Phi(S_i)$ and $P(y, t)$ lies in $\Phi(S_{i'})$ imply that one region in $\mathscr{R}(P)$ contains no hole ($\mathscr{R}(P)$ was defined before (3.3)).

Suppose that $\mathscr{D}(e, e') \neq \varnothing$. Since $(f^e-f^{e,e'})+(f^{e'}-f^{e,e'})=2-2\beta(\tau)=1$ and $f^{e,e'}>0$, we have $f^e>f^{e,e'}$ and $f^{e'}>f^{e,e'}$. So there are paths $P, P' \in \mathscr{L}$ such that $e \in P \not\ni e'$ and $e' \in P' \not\ni e$; let $P=s \cdots xy \cdots t$ and $P'=s' \cdots xz \cdots t'$. By the above arguments, (i) $P(s, x)$, $P'(s', x)$ lie in $\Phi(S_i)$; (ii) $P(y, t)$, $P'(z, t')$ lie in $\Phi(S_{i'})$; and (iii) there are $\Omega \in \mathscr{R}(P)$ and $\Omega' \in \mathscr{R}(P')$ which contain no hole. By (iii), $\Omega \subseteq \Psi_I$ and $\Omega' \subseteq \Psi_I$. Since $e' \notin P$, either e' is outside of Ω or e' lies in $\Omega - P$. In the latter case e' lies in the interior of the region $\Psi_I \cup I$, whence each path in $\mathscr{D}(e, e')$ belongs to $\mathscr{L}_I$ (by (18)(ii)). Now suppose that $e' \cap \Omega=\varnothing$ and $e \cap \Omega'=\varnothing$. In view of (i)–(ii) above $e=xy$ traverses a region $\Phi(T_j)$ and $e'=xz$ traverses a region $\Phi(T_{j'})$. Moreover, from the supposition one can deduce that $j \neq j'$; this means that the edges e, e' split $\Phi(S_i)$ into two parts, one containing $S_i \cap V(I) \neq \varnothing$ and the other containing $S_i \cap V(F) \neq \varnothing$ for some $F \in \mathscr{H}-\{I\}$. Then the graph $\langle S_i \rangle$ is not connected (taking into account that e, e' are consecutive in $E(x)$); a contradiction.

Finally, applying similar arguments to $\tau'=(u, x, u')$ we obtain $\mathscr{D}(u, u') \subseteq \mathscr{L}_I$. (Note that $\beta(\tau')=\beta(\tau)=\frac{1}{2}$.) ∎

Proof of Lemma 3.4. Let $C:=C_I$. For $h \in EG$ denote f_I^h by f_1^h and denote $f_J^h+f_K^h$ by f_2^h. Let E_1 (E_2) be the set of edges in $EG-C$ lying in Ψ_I (respectively, outside Ψ_I). Then the paths in $\mathscr{L}_J \cup \mathscr{L}_K$ use no edges in E_1, and the paths in $\mathscr{L}_I$ use no edges in E_2.

Consider a vertex x in C, and let h and h' be the edges in C incident to x. We say that x is an *i, j-vertex* if $|E(x) \cap E_1|=i$ and $|E(x) \cap E_2|=j$. From the fact that f_I is non-separating it follows that the elements of E_i occuring in $E(x)$ go in succession. Let $E(x)=(e_1, e_2, e_3, e_4)$.

CLAIM 1. *If x is a* 1, 1*-vertex, then $f_i^h=f_i^{h'}=1$ for some $i \in \{1, 2\}$.*

Proof. Let $h=e_1$ and $h'=e_3$. Then one of e_2, e_4 belongs to E_1 and the other to E_2, whence $\mathscr{D}(e_2, e_4)=\varnothing$. By (3.6), $f^h=f^{h'}=1$ and $\beta(e_1, x, e_2)=\frac{1}{2}$. Then, by (3.8), all paths in $\mathscr{D}(h)$ and $\mathscr{D}(h')$ belong to the same $\mathscr{L}_F$. ∎

CLAIM 2. *Let x be a 2, 0-vertex. If $f_i^h = 0$ for some $i \in \{1, 2\}$ then $f_2^{h'} = 0$.*

Proof. Observe that $f_2^u = 0$ for $u \in E(x) - \{h, h'\}$ (since $u \in E_1$). Therefore, $f_2^h = 0$ if and only if $f_2^{h'} = 0$. Now suppose that $f_1^h = 0$. Assuming that $h = e_1$ and $h' = e_2$, we have $f_2^{e_3} = 0$ (as $e_3 \in E_1$). Then $\mathscr{D}(e_1, e_3) = \varnothing$, whence $\beta(h, x, h') = \frac{1}{2}$ (by (3.6)). By (3.8), all paths in $\mathscr{P}(x)$ belong to the same $\mathscr{L}_F$. Since $f^{e_4} > 0$ (by (3.6)) and $f_2^{e_4} = 0$ (as $e_4 \in E_1$), only $F = I$ is possible. Hence, $f_2^{h'} = 0$. ∎

CLAIM 3. *Let x be a 0, 2-vertex. If $f_i^h = 0$ for some $i \in \{1, 2\}$ then $f_1^{h'} = 0$.*

Proof. Similar to that of Claim 2. ∎

Let $C = x_0 x_1 \cdots x_p$. Suppose that $f_1^i = 0$ or $f_2^i = 0$ for some $i \in \{1, ..., p\}$, where f^i stands for $f^{x_{i-1}x_i}$. Applying Claims 1–3 to the vertex x_i and the edges $h = x_{i-1}x_i$ and $h' = x_i x_{i+1}$, we obtain $f_1^{i+1} = 0$ or $f_2^{i+1} = 0$. This implies that $f_1^e = 0$ or $f_2^e = 0$ holds for each edge e in C.

Now suppose that $f_1^i > 0$ and $f_2^i > 0$ for all $i = 1, ..., p$. Then, by Claim 1, C has no 1, 1-vertices. Note also that C contains at least one 2, 0-vertex and one 0, 2-vertex (otherwise G would be not connected). So there are a 2, 0-vertex x and a 0, 2-vertex y such that x and y are adjacent in C. Let for definiteness $E(x) = (e_1, e_2, e_3, e_4)$, $E(y) = (u_1, u_2, u_3, u_4)$, $e_1 = u_1 = xy$, and $e_4, u_4 \in C$, see Fig. 4. Put $a_{ij} := f^{e_i, e_j}$ and $b_{ij} := f^{u_i, u_j}$. For $\tau = (e_4, x, e_1)$ and $\tau' = (u_4, y, u_1)$ we have

$$2 - a_{12} - a_{13} - a_{24} - a_{34} = 2\beta(\tau) \leqslant 1, \tag{21}$$

$$2 - b_{12} - b_{13} - b_{24} - b_{34} = 2\beta(\tau') \leqslant 1. \tag{22}$$

Note also that each path in $\mathscr{D}(e_i, e_1)$ for $i = 1, 2$ must pass through the edge u_4 (since $u_2, u_3 \in E_2$). Hence,

$$a_{12} + a_{13} + b_{24} + b_{34} \leqslant f^{u_4} \leqslant 1. \tag{23}$$

Similarly, each path in $\mathscr{D}(u_1, u_i)$ for $i = 1, 2$ must pass through e_4; therefore,

$$a_{24} + a_{34} + b_{12} + b_{13} \leqslant f^{e_4} \leqslant 1. \tag{24}$$

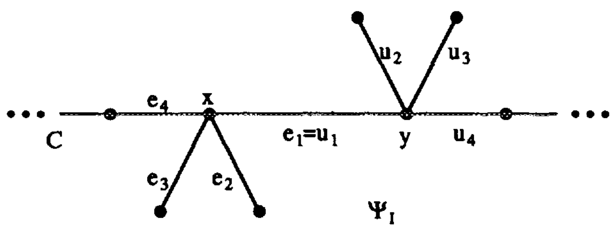


FIGURE 4

Summing up (21)–(24) we obtain $4 \leqslant 4$, so all the inequalities there hold with equality. Thus, $\beta(\tau) = \frac{1}{2}$. Then, by (3.8), all paths in $\mathscr{P}(x)$ belong to the same $\mathscr{L}_F$. This contradiction proves the lemma. ∎

Thus, one may assume that f is half-integral. Now we prove that (c, g) has an integral solution.

From the half-integrality of f it follows that, for a fork $\tau = (e, x, e')$:

(25) (i) if $f^{e,e'} > 0$ then $f^{e,e'} = \beta(\tau) = \frac{1}{2}$ and $f^e = f^{e'} = 1$ (as $\frac{1}{2} \geqslant \beta(\tau) \geqslant f^{e,e'} \geqslant f(P) \in \{0, \frac{1}{2}, 1\}$ for any $P \in \mathscr{D}(e, e')$);

(ii) if $f^e = 0$ then $\beta(\tau) = \frac{1}{2}$ and $f^{e'} = 1$ (as $\frac{1}{2} \geqslant \beta(\tau) \geqslant 1 - \frac{1}{2} f^e - \frac{1}{2} f^{e'}$).

For $F \in \mathscr{H}$ let G_F denote the subgraph of G contained in Ψ_F. Consider two holes, say I and J. By (18) we know that

$$\Psi_I \cap \Psi_J = C_I(J) \cap C_J(I) = G_I \cap G_J. \tag{26}$$

For a proper partition $\Pi(\sigma) = (T_1, T_2, S_1, S_2, S_3)$ of VG, the set S_i such that $S_i \cap V(I) \neq \varnothing$ and $S_i \cap V(J) \neq \varnothing$ is denoted by S_{IJ}.

3.9. *The circuits $C := C_I(J)$ and $C' := C_J(I)$ are disjoint (and similarly for the other pairs of holes).*

Proof. First of all observe that

(27) for each $x \in VG - T$ all paths in $\mathscr{P}(x)$ belong to the same set $\mathscr{L}_F$, $F \in \mathscr{H}$.

Indeed, let $E(x) = (e_0, e_1, e_2, e_3)$. If $f^{e_i, e_{i+1}} > 0$ or if $f^{e_i} = 0$ for some i then (27) follows from (25) and (3.8) (taking indices modulo 4). Otherwise, there are $P, P' \in \mathscr{P}(x)$ such that P contains e_0, e_2 while P' contains e_1, e_3. Then P, P' belong to the same $\mathscr{L}_F$ because of the regularity of f, and (27) follows as well.

Next, we assert that

(28) $f_J^e = f_K^e = 0$ for each $e \in C$ (and similarly for C').

For suppose that this is false. From (27) it follows that there is $F \in \mathscr{H} - \{I\}$ such that either (i) $f_F^e > 0$ for each $e \in C$ or (ii) $f^e = 0$ and $f_F^u > 0$ for some consecutive edges $e = xy$ and $u = xy'$ in C. Note that in both cases e belongs to $\mathrm{bd}(I)$, by definition of Ψ_I and $C_I(J)$. In case (ii), choose a fork $\tau = (e, x, e')$; then $\beta(\tau) = \frac{1}{2}$. In case (i), the facts that all edges of C are in $\mathrm{bd}(I)$, G is connected, and $T \cap V(I) \neq \varnothing$ imply that there is a fork $\tau = (e, x, e')$ such that $e = xy \in C$ while e' is outside of Ψ_F; then $f^{e'} = 0$ (by (27)), whence $\beta(\tau) = \frac{1}{2}$. For both cases, consider a metric $m = m_\sigma \in \mathscr{M}$ crucial for τ; let $\Pi(\sigma) = (T_1, T_2, S_1, S_2, S_3)$, $x \in S_1$ and $y, z \in S_2$, where z is the end of e' different form x. Since $x, y \in V(I)$, the pair $\{S_1, S_2\}$ coincides with $\{S_{IJ}, S_{IK}\}$ (by (4)). Next, obviously, $f^e + f^{e'} > f^{e,e'} = 0$; so there is a path $P \in \mathscr{L}$ containing exactly one of e, e'. By (3.7), P is shortest for m, whence P has one end in S_{IJ} and the other in S_{IK}. This means that $P \in \mathscr{L}_I$; a contradiction with the fact that $\mathscr{P}(x) \subseteq \mathscr{L}_F$. Thus, (28) is true.

Now suppose that C and C' have a common vertex x. By (27), $\mathscr{P}(x) \subseteq \mathscr{L}_F$ for some $F \in \mathscr{H}$; one may assume that $I \neq F$. Then for the edges e, e' in C incident to x, one has $f^e = f^{e'} = 0$ (by (28)). Hence e, e' are opposite edges in $E(x)$, and there is a path in L containing the other pair of opposite edges in $E(x)$. This contradicts the regularity of f. ∎

In what follows we assume that f_I and f_J are non-separating. By (3.9), the regions Ψ_I, Ψ_J, and Ψ_K are pairwise disjoint. Let $\mathscr{A} := \mathbb{R}^2 - (I \cup J \cup K \cup \Psi_I \cup \Psi_J \cup \Psi_K)$. If f_K is non-separating then $\mathscr{A}$ is the open region bounded by C_I, C_J, C_K; while if f_K is separating then $\mathscr{A}$ consists of two open regions one of which, say $\mathscr{A}_I$, is bounded by C_I and $C_K(I)$, and the other, $\mathscr{A}_J$, is bounded by C_J and $C_K(J)$ (see Fig. 3). Denote by $\mathscr{C}$ the set of distinct circuits among $C_F(F')$, $F, F' \in \mathscr{H}$. We say that $C \in \mathscr{C}$ *separates* holes F, F' if they are in different components of $\mathbb{R}^2 - C$.

Let B be the set of edges of G contained in $\mathscr{A}$, that is, $B = EG - (EG_I \cup EG_J \cup EG_K)$. Consider a fork $\tau = (e, x, e')$ with $e \in B$. Then $f^e = 0$, whence $\beta(\tau) = \frac{1}{2}$ and $f^{e'} = 1$. In particular, this implies that B is a matching. Let $m = m_\sigma \in \mathscr{M}$ be crucial for τ, and let $\Pi(\sigma) = (T_1, T_2, S_1, S_2, S_3)$. From (3.7) and the fact that $\mathscr{D}(e, e') = \varnothing$ it follows that

(29) (i) $f^u = 1$ for any $u \in EG - \{e\}$ such that $m(u) > 0$;

(ii) each path in $\mathscr{L}$ is shortest for m.

Consider some S_i. By (4), $\Phi(S_i)$ meets the boundaries of exactly two holes, say F and F', and the part of bd(F) (respectively, bd(F')) lying in $\Phi(S_i)$ is connected. In addition, from (29)(ii) we observe that: (i) each path in $\mathscr{L}_F$ of $\mathscr{L}_{F'}$ crosses the boundary of $\Phi(S_i)$ at most once; and (ii) each path in $P \in \mathscr{L}_{F''}$ (where $\mathscr{H} = \{F, F', F''\}$) either does not meet $\Phi(S_i)$, or it crosses its boundary twice (in the latter case $f_{F''}$ is separating, P connects T_1 and T_2, and it meets neither $\Phi(S_{i-1})$ nor $\Phi(S_{i+1})$). Using these arguments, one can show (e.g., by induction on $|\mathscr{L}|$) that for any $C \in \mathscr{C}$:

(30) (i) if C does not separate F and F' then C does not meet $\Phi(S_{FF'})$;

(ii) if C meets $\Phi(S_{FF'})$ then the part of C contained in $\Phi(S_{FF'})$ is connected.

Now we finish the proof of Theorem 1 as follows:

(i) Suppose that f_K is separating. Then the circuit $C := C_K(I)$ does not separate J and K. Choose an edge $e = xy$ with x in C and y in C_I

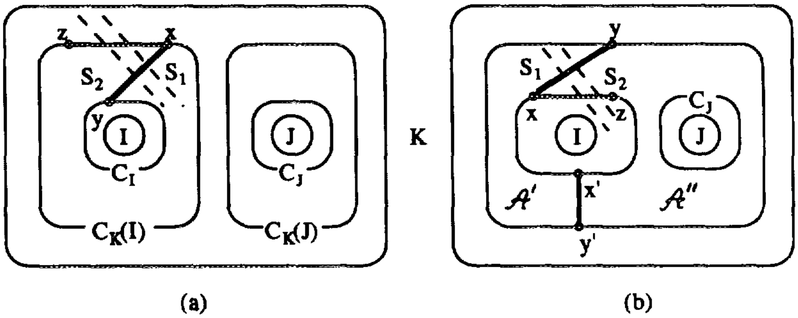


FIGURE 5

(e exists because G is connected). Consider a fork $\tau=(e, x, e')$; then the edge $e'=xz$ is in C. Let $m=m_\sigma \in \mathscr{M}$ be crucial for τ, let $\Pi(\sigma)=(T_1, T_2, S_1, S_2, S_3)$, and let $x \in S_1$ and $y, z \in S_2$, see Fig. 5a. By (29), there is a path $P \in \mathscr{L}$ containing e' and shortest for m. Then one end of P is in S_1 and the other in S_2. Since e' is in Ψ_K, P belongs to $\mathscr{L}_K$. Therefore, $S_i \cap V(K) \neq \varnothing$ for $i=1, 2$, whence $\{S_1, S_2\}=\{S_{IK}, S_{JK}\}$ (by (4)). Since C meets both S_1 and S_2 (as $x \in S_1$ and $z \in S_2$), we conclude that C meets S_{JK}. But C does not separate J and K, a contradiction with (30)(i). Thus, all f_I, f_J, f_K are non-separating.

(ii) Suppose that two edges $u=xy$ and $u'=x'y'$ in B connect the same pair of circuits in $\mathscr{C}$, say C_I and C_K. Let x, x' be in C_I, and y, y' be in C_K. The edges u and u' split $\mathscr{A}$ into two regions $\mathscr{A}'$ and $\mathscr{A}''$; the inner boundary of one of them, say $\mathscr{A}''$, is C_J; see Fig. 5b. Let $e=xz$ be the edge in C_I such that e belongs to the outer boundary of $\mathscr{A}''$. Consider $m=m_\sigma \in \mathscr{M}$ crucial for $\tau=(u, x, e)$; let $\Pi(\sigma)=(T_1, T_2, S_1, S_2, S_3)$, $x \in S_1$ and $y, z \in S_2$. Since e belongs to a path in $\mathscr{L}_I$, $\{S_1, S_2\}=\{S_{IJ}, S_{IK}\}$. Moreover, $S_2=S_{IK}$ because S_2 meets both C_I (at z) and C_K (at y). Hence, $S_1=S_{IJ}$. Let X be the set of vertices of C_I that are in S_1. We know that: S_1 does not meet C_K (by (30)(i)); $S_1 \cap V(J) \neq \varnothing$; the graph $\langle S_1 \rangle$ is connected; $z \notin S_1 \ni x$; and the elements of X go in C_I in succession (by (30)(ii)). These facts imply that X must contain x'. Thus, $x' \in S_1$ and $y' \notin S_1$, whence $m(u')>0$. But $f^{u'}=0$ (as $u' \in B$), a contradiction with (29)(i).

It remains to consider the case when each two circuits among C_I, C_J, C_K are connected by at most one edge in G. Then each pair of these circuits are connected by exactly one edge because G is connected and $|\delta(VG_F)|$ is even for each hole F. For $F \in \mathscr{H}$ add to G_F the edges from B with both ends in C_F, forming the graph G'_F. We assert that problem (G'_I, U_I) (and similarly, (G'_J, U_J) and (G'_K, U_K)) has a solution, where U_F is the set of pairs $\{s, t\} \in U$ with $s, t \in V(F)$. Let e connect C_I and C_J, and e' connect C_I and C_K.

Consider $G' := G'_I$ and $U' := U_I \cup \{\{x, y\}\}$, where x, y are the ends of e, e' in C_I. It suffices to show that (G', U') has a solution. Obviously, G' and U' satisfy (1). By Okamura's theorem, (G', U') has a solution if

$$b(X) := |\delta^{G'}X| - |\{\{s, t\} \in U' \mid X \text{ separates } s \text{ and } t\}| \geqslant 0 \qquad (31)$$

holds for any $X \subset VG'$. Inequality (31) follows from (2) if X does not separate x and y. Let $X \subset VG'$ separate x and y. Then $|\delta^{G'}X| = |\delta^G X| - 1$, and X separates exactly one pair more in U' than in U. Hence,

$$\Delta := |\delta^G X| - |\{\{s, t\} \in U \mid X \text{ separates } s \text{ and } t\}| = b(X) + 2.$$

Suppose that $b := b(X) < 0$. Then $b \leqslant -2$ (as b is even), whence $\Delta = 0$ (as $\Delta \geqslant 0$). This means that f, being a solution of (c, g), saturates all edges in $\delta^G X$ (since $c(X) = g(X)$). But $\delta^G X$ contains e or e', and $f^e = f^{e'} = 0$, a contradiction. This completes the proof of Theorem 1.

4. The Case of Four Holes

We give a counterexample (G, U) to the statement (1.1) for $|\mathscr{H}| = 4$. Figure 6 illustrates a planar graph G' (whose edges are drawn by the solid lines) and a set U of pairs in VG' (indicated by dotted lines); here $\mathscr{H} = \mathscr{F}_{G'} = \{I, J, K, O\}$. The graph G is obtained from G' by replacing each edge of G' by two parallel edges. Then G and U satisfy (1). Instead of considering G, it is convenient to mean that G' has the capacity $c(e) = 2$ for each its edge e.

Each $\{s', t'\} \in U$ belongs to the boundary of exactly one hole F of G'; let $\mathscr{P}(s', t')$ denote the pair of simple $s' - t'$ paths going along the boundary of

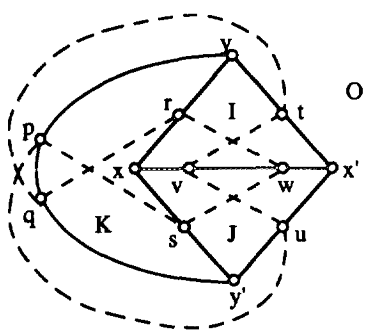


Figure 6

F. The problem (c, g) (with g to be all-units on U) has a solution f, defined by $f(P)=f(P')=\frac{1}{2}$ for $\{P, P'\}=\mathscr{P}(s', t')$, $\{s', t'\}\in U$.

In order to prove that (c, g) has no integral solution consider the length function l on EG' such that $l(e)=2$ for $e=pq$, vw and $l(e)=1$ for the other edges e in EG'. Then for any $\{s', t'\}\in U$, we have $\text{dist}_l(s', t')=4$ and there are exactly two $s'-t'$ paths shortest for l, namely, the paths in $\mathscr{P}(s', t')$. In addition, one can see that

$$cl=\sum (\text{dist}_l(s', t') \mid \{s', t'\}\in U).$$

This implies that any f' solving (c, g) saturates each edge of G', and each path P with $f'(P)>0$ is shortest for l. Thus, to construct an integral solution for (c, g) one has to choose one path in each $\mathscr{P}(s', t')$ ($\{s', t'\}\in U$) in such a way that any edge of G' should belong to exactly two of these paths.

Consider possible choices of paths for the pairs $\{r, w\}$, $\{t, v\}$, $\{s, w\}$, $\{u, v\}$. Up to the symmetry of I and J, there are only two possibilities: (i) $P_1 := rxvw$, $P_2 := tx'wv$, $P_3 := sy'ux'w$, $P_4 := uy'sxv$; and (ii) $Q_1 := rxvw$, $Q_2 := tyrxv$, $Q_3 := sy'ux'w$, $Q_4 := ux'wv$. In case (i), the edges sy' and uy' are covered twice (by P_3 and P_4); therefore, the edge qy' cannot be saturated. Similarly, in case (ii), xs cannot be saturated. This, (c, g) has no integral solution.

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