Minimum $(2,r)$-metrics and integer multiflows *

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Abstract. Let $H = (T, U)$ be a connected graph. A *T-partition* of a set $V \supseteq T$ is a partition of V into subsets, each containing exactly one element of T.

We start with the following problem $(*)$: given a multigraph $G = (V, E)$ with $V \supseteq T$, find a T-partition Π of V that minimizes the sum of products $d(s,t)n(s,t)$ over all $s, t \in T$. Here $d(s,t)$ is the distance from s to t in H and $n(s,t)$ is the number of edges of G between the sets in Π that contain s and t. When the graph H is complete, $(*)$ turns into the minimum multiway cut problem, which is known to be NP-hard even if $|T| = 3$. On the other hand, when H is the complete bipartite graph $K_{2,r}$ with parts of 2 and $r = |T| - 2$ nodes, $(*)$ is specialized to be the minimum $(2,r)$-metric problem, which can be solved in polynomial time.

We prove that the multicommodity flow problem dual of the minimum $(2,r)$-metric problem has an integer optimal solution whenever G is *inner Eulerian* (i.e., the degree of each node in $V - T$ is even), and such a solution can be found in polynomial time.

Another nice property of $K_{2,r}$ is that, independently of G, the optimum objective value in $(*)$ is the same as that in its fractional relaxation. We call a graph H with a similar property *minimizable* and give a description of the minimizable graphs in polyhedral terms. Finally, we show that every tree is minimizable.

Key words: metric, cut, multiway cut, multicommodity flow.

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1. Introduction

Let $G = (V, E)$ be an undirected graph, $T \subseteq V$ a subset of nodes, and $\mu : T \times T \to \mathbb{Z}_+$ a symmetric function, i.e., $\mu(s,t) = \mu(t,s)$ for $s,t \in T$. We allow multiple edges in G and usually assume that G is described by use of its edge multiplicity function $c = c^G$ which indicates how many edges connect nodes x and y in G, for all $x, y \in V$ (this is important for algorithmic aspects). A *T-partition* is a partition of V into $|T|$ subsets X_t, $t \in T$, each containing exactly one element of T, namely, $t \in X_t$. Consider the *minimum T-partition problem*:

(1.1) Find a T-partition $\{X_t : t \in T\}$ of V that minimizes the sum of products $\mu(s,t)n(s,t)$ over all $s,t \in T$, where $n(s,t)$ is the number of edges of G between X_s and X_t.

We deal with the special case of (1.1) in which μ is the *distance function* d^H of a connected graph $H = (T, U)$ on T, i.e., for $s,t \in T$, $d^H(s,t)$ is the minimum length (number of edges) of a path between s and t in H. In particular, $\mu(s,t) = 0$ if $s = t$, and we may assume that G has no loops. When H is the complete graph K_p with $p = |T|$ nodes, (1.1) is further specialized to be the *minimum multiway cut problem.* In other words, it is required to minimize the number of edges of G connecting different sets in a T-partition. This problem is known to be NP-hard even if $p = 3$ [2]. On the other hand, if $p = 2$, the problem is efficiently solvable as being the minimum cut problem for which plenty of polynomial and strongly polynomial time algorithms are known (assuming that G is given via c as above).

Another interesting special case arises when H is the complete bipartite graph $K_{2,r}$ with parts of 2 and $r = |T| - 2$ nodes. It turns out that in this case (1.1) can be solved in strongly polynomial time [4]. This fact is also a consequence of the property that (1.1) with $\mu = d^{K_{2,r}}$ is, in essense, equivalent to its fractional relaxation. The property of such a kind is important for us in this paper and we are going to explain it in more details, starting with some terminology and notations.

By a *metric* on a finite set V we mean a function $m : V \times V \to \mathbb{R}_+$ which is symmetric and satisfies

(i) $m(x,x) = 0$ for $x \in V$; and

(ii) $m(x,y) + m(y,z) \geq m(x,z)$ for $x,y,z \in V$ (*triangle inequalities*).

Note that we allow zero values $m(x,y)$ for distinct x and y (i.e., in fact we deal with *semi-metrics*). Because of (i) and the symmetry we may assume that m is given on the set of edges E_V of the complete undirected graph on V, using notation $m(e)$ or $m(xy)$ for $e = xy \in E_V$ and, when needed, letting by definition $m(xx) = 0$.

Given a graph $H = (T, U)$, a metric m on V is called an *extension* of H (or d^H) to V if m coincides with d^H within T, and a 0-*extension* if there is a T-partition $\{X_t : t \in T\}$ of V such that $m(xy) = d^H(st)$ for all $s, t \in T$, $x \in X_s$ and $y \in X_t$. 0-extensions of K_2 and $K_{2,r}$ are called *cut metrics* and $(2, r)$-*metrics*, respectively. For $\mu = d^H$, (1.1) turns into the *minimum* 0-*extension problem*:

(1.2) Find a 0-extension m of H to V with $c \cdot m$ as small as possible.

Here and later on we denote by $a \cdot b$ the inner product $\sum(a(e)b(e) : e \in S)$ of functions a and b within the common part S of their domains. The *fractional relaxation* of (1.2) is stated as follows:

(1.3) Find an extension m of H to V with $c \cdot m$ as small as possible.

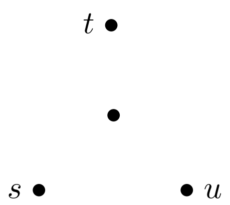

Fig. 1

Let $\tau = \tau(G, H)$ and $\tau^* = \tau^*(G, H)$ denote the minima of $c \cdot m$ in (1.2) and (1.3), respectively. Since a 0-extension is an extension, we have $\tau \geq \tau^*$. In general, this inequality may not be equality. E.g., if G is as in Fig. 1, $T = \{s, t, u\}$ and H is the complete graph on T, then $\tau = 2$ while $\tau^* = 3/2$. The simplest case with $\tau = \tau^*$ arises when $H = K_2$ (this is a reformulation of the fact that the minimum cut problem can be stated as an integer linear program with a totally unimodular matrix; see, e.g., [3]). A similar property is true for $H = K_{2,r}$.

Theorem 1.1 [4]. *If $H = K_{2,r}$ then $\tau = \tau^*$.*

As mentioned above, there exists an efficient algorithm for solving the minimum $(2, r)$-metric problem (i.e., (1.2) with $H = K_{2,r}$). The existence of such an algorithm is provided by Theorem 1.1 and a general observation, as follows. Let us say that a graph $H = (T, U)$ is *minimizable* if the equality $\tau = \tau^*$ holds for any graph $G = (V, E)$ with $V \supseteq T$. E.g., K_2 and $K_{2,r}$ are minimizable while K_p and $K_{p,r}$ $(p, r \geq 3)$ are not. For a minimizable H, (1.2) can be solved in strongly polynomial time. Indeed, for an arbitrary H, (1.3) can be written as the linear program:

(1.4) $$\text{minimize} \quad c \cdot m \quad \text{subject to}$$
$$m \geq 0;$$
$$m \text{ satisfies the } (|V| - 2)\binom{|V|}{2} \text{ triangle inequalities;}$$

$$m(st) = d^H(st) \quad \text{for } s,t \in T.$$

Since the constraint matrix M in (1.4) consists of $O(|V|^3)$ rows and $O(|V|^2)$ columns and all entries of M are 0,+1 or −1, a version of the ellipsoid method from [8] can be applied to find τ^* in strongly polynomial time. Now if we know that H is minimizable, (1.2) is reduced in an obvious way to comparing $\tau^*(G,H)$ and $\tau^*(G',H)$ for a sequence of graphs G', each obtained from G by sticking some nodes in $V-T$ to nodes in T; clearly it suffices to test at most $|V-T||T|$ graphs G'. (Note that a faster algorithm for $H = K_{2,r}$ in [4] applies the ellipsoid method only once.)

Next we discuss duality aspects for (1.2) and (1.3). They come up by analogy with the classic duality between the minimum cut and maximum flow problems. In a general case, (1.3) is dual to a certain multicommodity flow problem, as follows. A simple path in the complete graph (V, E_V) connecting different nodes in T is called a *T-path.* By a *multicommodity flow*, or, simply, a *multiflow*, for V,T is a pair $f = (\mathcal{P}, \lambda)$ consisting of T-paths $P_1, \ldots, P_k$ along with nonnegative real numbers $\lambda_1, \ldots, \lambda_k$. Define

(1.5) $$f^e = \sum(\lambda_i \,:\, P_i \text{ contains } e) \quad \text{for } e \in E_V;$$
$$f_{st} = \sum(\lambda_i \,:\, P_i \text{ connects } s \text{ and } t) \quad \text{for } s,t \in T.$$

Considering $c = c^G$ as an edge capacity function, we call f *c-admissible* if

(1.6) $$f^e \leq c(e) \qquad \text{for all } e \in E_V.$$

The value of f with respect to H, or the *H-value* of f, is $\sum(d^H(st)f_{st} \,:\, s,t \in T)$, denoted by $\langle H, f \rangle$. The kind of multiflow problems we deal with is:

(1.7) Maximize $\langle H, f \rangle$ among all c-admissible multiflows f for V,T.

Let ν^* denote the maximum of $\langle H, f \rangle$ in (1.7), and ν the maximum of $\langle H, f \rangle$ if the only *integer* multiflows f (i.e., with all λ_i's integer) are allowed. Clearly (1.7) is a linear program, and assigning dual variables $m(e)$ to the constraints in (1.6), we observe that the program dual of (1.7) consists in minimizing $c \cdot m$ over all $m : E_V \to \mathbb{R}_+$ such that for each $s,t \in T$, the m-length of each path P connecting s and t is at least d^H. Now decreasing, if needed, m on some edges, we obtain a metric feasible to (1.3). This implies $\nu^* = \tau^*$, so we may think of (1.7) and (1.3) as a pair of mutually dual programs. This minimax relation between metrics and multiflows was originally revealed by Lomonosov [6].

Obviously, $\nu \leq \nu^*$, and this inequality may be strict. E.g., for the above example with $H = K_3$ and G depicted in Fig. 1, $\nu = 1$ while $\nu^* = 3/2$. Nevertheless, Lovász [7] and Cherkassky [1], independently, proved that if $H = K_p$ and G is inner Eulerian, then $\nu = \nu^*$. Here G is called *inner Eulerian* if each node in $V - T$ is incident to an even number of edges of G. In case $H = K_{2,r}$ the equality $\nu = \nu^*$ may not hold either. In this paper we prove the following theorem.

Theorem 1.2. *If $H = K_{2,r}$ and G is inner Eulerian, then $\nu = \nu^*$.*

Theorems 1.1 and 1.2 imply $\nu = \tau$ for $H = K_{2,r}$ and an inner Eulerian G.

This paper is organized as follows. Theorem 1.2 is proved in Section 2. The proof is based on a splitting-off method and provides a strongly polynomial algorithm to find an integer optimal multiflow in the inner Eulerian case. The details of this algorithm are described in Section 3. It should be noted that the algorithm relies on the ellipsoid method to certify the feasibility of the splitting-off operations that we apply. Finally, concluding Section 4 returns us to a general case of H, describes the minimizable graphs in polyhedral terms and presents a simple operation on graphs to construct more minimizable graphs. This will show that the set of minimizable graphs is rather large; in particular, it includes every tree.

2. Proof of the theorem

We show that if $H = K_{2,r}$ and G is inner Eulerian, then $\nu = \tau$. The proof borrows some ideas from [4] and relies on certain transformations of the function $c = c^G$. In order to distinguish between the values of ν (τ, ν^*, τ^*) for different capacity functions we use notation $\nu(c')$ (respectively, $\tau(c'), \nu^*(c'), \tau^*(c')$), where c' is a function on E_V under consideration. A function $c' : E_V \to \mathbb{Z}_+$ is called inner Eulerian if $\sum(c'(xy) \,:\, y \in V)$ is even for each $x \in V - T$.

Let $\mathcal{E}$ denote the set of 0-extensions of H to V. A metric $m \in \mathcal{E}$ is called *tight* for c if $c \cdot m = \tau(c)$; the set of tight m's is denoted by $\mathcal{T}(c)$. Let

$$\eta(c) = \sum_{x \in V-T} \sum_{y \in V - \{x\}} c(xy).$$

We use induction, assuming that the equality $\nu(c') = \tau(c')$ holds for each inner Eulerian c' on E_V such that either $|\mathcal{T}(c')| > |\mathcal{T}(c)|$, or $|\mathcal{T}(c')| = |\mathcal{T}(c)|$ and $\eta(c') < \eta(c)$ (note that $|\mathcal{T}(c)| \le |\mathcal{E}|$ and $\mathcal{E}$ is finite for V fixed). The base case $\mathcal{T}(c) = \mathcal{E}$ together with $\eta(c) = 0$ is easy (in fact the first condition is equivalent to the second one when $r > 0$). Indeed, $\eta(c) = 0$ implies $c \cdot m = c \cdot d^H$ for any 0-extension m, and the multiflow f formed by the elementary paths P_{uv} that consist of of one edge connecting distinct $u, v \in T$, along with weights $\lambda_{uv} = c(uv)$, has the H-value equal to $c \cdot d^H$, whence $\nu(c) = \tau(c)$. So, in the sequel we assume that $\eta(c) > 0$.

Consider $x \in V - T$ for which the set $Q(x) = \{y \in V \,:\, c(xy) > 0\}$ is nonempty.

We may assume that $|Q(x)| \geq 2$. Indeed, if $Q(x)$ consists of a single element y, then $c(xy) \geq 2$ (as c is inner Eulerian). Decrease $c(xy)$ by 2. Then the resulting function c' is nonnegative and inner Eulerian. Moreover, it is easy to see that there is an $m \in \mathcal{E}$ such that $c' \cdot m = \tau(c')$ and both x and y belong to the same set in the T-partition of V corresponding to m (i.e., $m(xy) = 0$). Then $c' \cdot m = c \cdot m$, implying $\tau(c') = \tau(c)$. Obviously, $\mathcal{T}(c) \subseteq \mathcal{T}(c')$ and $\eta(c) > \eta(c')$, and the result follows by induction.

Let Φ be the set of pairs of distinct elements of $Q(x)$. The *splitting-off operation* applied to a pair $\{y, z\} \in \Phi$ transforms c as follows:

$$
\begin{aligned}
c'(e) &= c(e) - 1 \quad \text{for } e = xy, xz, \\
&= c(e) + 1 \quad \text{for } e = yz, \\
&= c(e) \quad \text{for } e \in E_V - \{xy, xz, yz\}.
\end{aligned}
\tag{2.1}
$$

Clearly c' is nonnegative and inner Eulerian. For any metric m on V, $c \cdot m - c' \cdot m = m(xy) + m(xz) - m(yz) \geq 0$. Therefore, $\tau(c') \leq \tau(c)$. We say that $\{y, z\}$ is *feasible* if $\tau(c') = \tau(c)$. In this case the relations $c \cdot m \geq c' \cdot m \geq \tau(c') = \tau(c)$ for an arbitrary $m \in \mathcal{E}$ imply that any metric from $\mathcal{T}(c)$ remains tight for c' too; therefore, $\mathcal{T}(c) \subseteq \mathcal{T}(c')$. Also $\eta(c') = \eta(c) - 1$. By induction there exists a c'-admissible integer multiflow f' with $\langle H, f' \rangle = \tau(c')$. Now f' can be transformed in an obvious way into a c-admissible integer multiflow f of the same H-value. Hence, $\nu(c) \geq \langle H, f \rangle = \tau(c)$, which implies $\nu(c) = \tau(c)$, as required.

Our aim is to show that there exists at least one feasible pair in Φ, from which the theorem will follow by the above argument. Let $\{s_1, s_2\}$ and $\{t_1, \ldots, t_r\}$ be the parts of $H = K_{2,r}$.

Claim 1. *For any $m \in \mathcal{E}$, $c \cdot m - \tau(c)$ is even.*

Proof. Consider the T-partition $\{S_1, S_2, T_1, \ldots, T_r\}$ of V corresponding to m, where $s_i \in S_i$ and $t_j \in T_j$. Let ρ be the cut metric corresponding to the cut separating $X = S_1 \cup S_2$ from $V - X$, i.e., $\rho(xy) = 1$ if $|\{x, y\} \cap X| = 1$, and 0 otherwise. Then $m + \rho$ takes value 0 or 2 on each edge, whence $c \cdot (m + \rho)$ is even. Now the claim follows from the fact that for each cut metric ρ' that corresponds to a cut in G separating $\{s_1, s_2\}$ from $\{t_1, \ldots, t_r\}$, the number $c \cdot \rho - c \cdot \rho'$ is even (because c is inner Eulerian). •

Consider c' as in (2.1) for some $\{y, z\} \in \Phi$.

Claim 2. *For each $m \in \mathcal{E}$, $\Delta = c \cdot m - c' \cdot m$ equals 0, 2 or 4. Moreover, if $\Delta = 4$ then $m(xy) = m(xz) = 2$ and $m(xz) = 0$ (and therefore, both y and z belong to the same member of the T-partition of V corresponding to m).*

Proof. We have $\Delta = m(xy) + m(xz) - m(yz) \geq 0$. Observe that the length of any closed path with respect to a $(2,r)$-metric is even. This implies that Δ is even. Next, $m(uv) \leq 2$ for any $u, v \in V$. Hence, $\Delta \in \{0, 2, 4\}$. If $\Delta = 4$ then the only possible case is $m(xy) = m(xz) = 2$ together with $m(yz) = 0$. •

The infeasibility of $\{y, z\} \in \Phi$ is equivalent to the existence of $m \in \mathcal{E}$ such that $c' \cdot m$ is strictly less than $\tau(c)$. From Claims 1 and 2 it follows that

(2.2) if $\{y, z\} \in \Phi$ is infeasible, then for each $m \in \mathcal{E}$ with $c' \cdot m < \tau(c)$, either

(i) m is tight and $m(xy) + m(xz) - m(yz) > 0$, or

(ii) $c \cdot m = \tau(c) + 2$, $c' \cdot m = \tau(c) - 2$ and $m(xy) + m(xz) - m(yz) = 4$.

A metric m as in (2.2)(ii) is called *critical* for c and $\{y, z\}$.

In what follows we assume that each pair in Φ is infeasible and will attempt to come to a contradiction. First we show that there exists $\{y, z\} \in \Phi$ for which the only second alternative in (2.2) takes place.

By Theorem 1.1, $\tau(c) = \tau^*(c) = \nu^*(c)$. So, there is a c-admissible multiflow $f = (P_1, \ldots, P_k; \lambda_1, \ldots, \lambda_k)$ with $\langle H, f \rangle = \tau(c)$; we assume that $\lambda_i > 0$ for $i = 1, \ldots, k$. An edge e is called *saturated* by f if $f^e = c(e)$ (cf. (1.6)). Let q_i be the pair of end nodes of P_i. For a path P, $m(P)$ stands for the sum of $m(e)$'s over the edges e of P.

Claim 3. *Let $\{y, z\} \in \Phi$ and $m \in \mathcal{T}(c)$. Then:*

(i) *if $m(xy) > 0$ then xy is saturated by f; and similarly for xz;*

(ii) *each path P_i is shortest for m, i.e., $m(P_i) = d^H(q_i)$.*

Proof. (i) and (ii) immediately follow from consideration of the complementary slackness conditions for (1.3) and (1.7). More precisely,

$$\nu^*(c) = \sum_{i=1}^{k} \lambda_i d^H(q_i) \leq \sum_{i=1}^{k} \lambda_i m(P_i)$$
$$= \sum_{e \in E_V} f^e m(e) \leq \sum_{e \in E_V} c(e) m(e) = \tau^*(c).$$

Since $\nu^*(c) = \tau^*(c)$, equality holds throughout, whence (i) and (ii) follow. •

Claim 3 gives the following property: for $\{y, z\} \in \Phi$, if xy is not saturated by f or there is a path in f that contains both xy and xz, then $m(xy) + m(xz) = m(yz)$ for any $m \in \mathcal{T}(c)$. This easily implies that there is $\{y, z\} \in \Phi$ for which no metric as in (2.2)(i) exists. We fix some of such $\{y, z\}$'s. By (2.2), there is a metric critical for c and $\{y, z\}$.

Consider the capacity function $\widetilde{c} = 2c$. We have $\tau(\widetilde{c}) = 2\tau(c)$. Furthermore, by (2.2) and the choice of $\{y, z\}$, any metric $m \in \mathcal{E}$ with $m(xy) + m(xz) > m(yz)$ satisfies $\widetilde{c} \cdot m \geq \tau(\widetilde{c}) + 4$. Hence, $\{y, z\}$ becomes feasible for $\widetilde{c}$. This implies that the function $\widetilde{c}'$ formed from $\widetilde{c}$ by the splitting-off operation with respect to $\{y, z\}$ satisfies $\tau(\widetilde{c}') = \tau(\widetilde{c}) = 2\tau(c)$. Let m be critical for c and $\{y, z\}$. Then

$$\widetilde{c} \cdot m = \tau(\widetilde{c}) + 4 \qquad \text{and} \qquad \widetilde{c}' \cdot m = \tau(\widetilde{c}).$$

Thus, $\mathcal{T}(\widetilde{c}')$ strictly includes $\mathcal{T}(\widetilde{c}) = \mathcal{T}(c)$. Obviously, $\widetilde{c}'$ is inner Eulerian. By induction there is an integer $\widetilde{c}'$-admissible multiflow h with $\langle H, h\rangle = \tau(\widetilde{c}')$. We transform h in an obvious way into a $\widetilde{c}$-admissible integer multiflow $g = (P_1, \ldots, P_k;\ \lambda'_1, \ldots, \lambda'_k)$. Define f to be the multiflow formed by the same paths P_i and the numbers $\lambda_i = \lambda'_i/2$, $i = 1, \ldots, k$. Then f is c-admissible and *half-integer*, and $\langle H, f\rangle = \tau(c)$. Repeating paths in f, if needed, we may assume that each λ_i is $1/2$.

For two nodes u and v in a path P, *truncating* P at $\{u, v\}$ is an operation that replaces in P the part between u and v by the edge uv. Consider a path P_i that passes through x (such a path must exist, otherwise each pair in Φ is, obviously, feasible). For definiteness let P_i use edges $e = xy$ and $e' = xz$.

Claim 4. *The edges e and e' are saturated by f.*

Proof. Consider $m \in \mathcal{E}$ critical for c and $\{y, z\}$. As above, let $\widetilde{c}'$ be obtained from $\widetilde{c}$ by the splitting-off operation with respect to $\{y, z\}$. Let h be the multiflow obtained from g as above by truncating P_i at $\{y, z\}$. Since $\lambda'_i = 2\lambda_i = 1$, h is $\widetilde{c}'$-admissible. Also $\langle H, h\rangle = \tau(\widetilde{c}')$ and m is tight for $\widetilde{c}'$. By Claim 3 applied to $\widetilde{c}', h, m, e, e'$, we have $h^e = \widetilde{c}'(e)$ and $h^{e'} = \widetilde{c}'(e')$. This implies that $f^e = c(e)$ and $f^{e'} = c(e')$. •

By Claim 4 there are paths P_l and P_q $(l, q \neq i)$ which contain e and e', respectively. Let a_l (b_l) and a_q (b_q) be the first (respectively, last) node in P_l and P_q, respectively. We may assume that a_l, y, x, b_l follow in this order in P_l, and a_q, z, x, b_q follow in this order in P_q.

Claim 5. $a_l = a_q$.

Proof. Consider a metric $m \in \mathcal{E}$ critical for $\{y, z\}$ and the partition $\Pi = \{S_1, S_2, T_1, \ldots, T_r\}$ of V corresponding to m, where $s_\alpha \in S_\alpha$ and $t_\beta \in T_\beta$. Let $\widetilde{c}'$ be obtained from $\widetilde{c}$ by the splitting-off operation with respect to $\{y, z\}$, and let h be the multiflow obtained from g by truncating P_i at $\{y, z\}$. Then h is $\widetilde{c}'$-admissible, $\langle H, h\rangle = \tau(\widetilde{c}')$, and m is tight for $\widetilde{c}'$. By Claim 2 applied to $\widetilde{c}, \widetilde{c}', m, x, y, z$, either $y, z \in S_j$ and $x \in S_{j'}$, or $y, z \in T_j$ and $x \in T_{j'}$ for distinct j, j'. Assume the former, the other case is similar. By (ii) in Claim 3, the path P_l is shortest for m. Since $d^H(a_l b_l) \leq 2$ and $m(xy) = d^H(s_{j'} s_j) = 2$, we observe that $m(e)$ must be zero for each edge of P_l different from xy. Hence, a_l and

y belong to the same set in Π, i.e., $a_l \in S_j$. Similar arguments for P_q yield $a_q \in S_j$. Since S_j contains exactly one element of T, namely, s_j, we conclude that $a_l = a_q$. •

Now we finish the proof as follows. We assume that f is chosen so that f is half-integer, $\langle H, f \rangle = \tau(c)$ and $\sum(f^e : e \in E_V)$ is as small as possible. Also we may assume that for each path P_i in f all inner nodes of P_i are not in T (otherwise split P_i into two T-paths, which does not decrease the H-value), and that some path P_i has at least two edges. Let y, x, z be the first, second and third node in P_i, respectively; then $x \in V - T$. Let P_l, a_l and P_q, a_q be defined as above for our y, x, z. By Claim 5, $a_l = a_q = y$ (as $y \in T$). So, P_i and P_q have the same first nodes and go through the edge xz in opposite directions. Therefore, we can replace P_i and P_q by two paths which have the same first node y, have the last nodes as in P_i and P_q, use merely the edges from these paths and release the edge xz. This contradicts the minimality of $\sum(f^e : e \in E_V)$ and completes the proof of Theorem 1.2. ••

3. Algorithm

The splitting-off techniques developed in the proof of Theorem 1.2 gives rise to an algorithm for finding an integer c^G-admissible multiflow f with $\langle H, f \rangle = \tau(G, H)$ for $H = K_{2,r}$ and an inner Eulerian G. When G is not inner Eulerian, we can apply the algorithm to the capacity function $2c$ to obtain a half-integer optimal solution for G.

The algorithm consists of two *stages*. The first stage consists of $|V - T|$ *iterations*, each of which treats a node $x \in V - T$. At a current *step* of the iteration for x, we choose a pair $\{y, z\} \in V - \{x\}$ with $b = \min\{c(xy), c(xz)\} > 0$ (for the current c) and finds the maximum $\alpha \in \mathbb{Z}_+$ such that $\alpha \leq b$ and $\tau(c') = \tau(c)$, where c' is defined by

$$
\begin{aligned}
(3.1) \qquad c'(e) &= c(e) - \alpha \quad \text{for } e = xy, xz, \\
&= c(e) + \alpha \quad \text{for } e = yz, \\
&= c(e) \quad \text{for } e \in E_V - \{xy, xz, yz\}
\end{aligned}
$$

(i.e., c' is obtained by performing splitting-off operation (2.1) α times for the same $\{y, z\}$). Then we make c' the new current c, choose a new pair $\{y', z'\}$, and so on. We need not consider the same pair $\{y, z\}$ twice during the iteration because, after the first application of splitting-off operation (3.1) to $\{y, z\}$, the corresponding number α for the new function c becomes zero and it will remain zero up to termination of the iteration. Since the problem for each current c has an integer optimal solution, the iteration always terminates, after $O(|V|^2)$ steps, with the situation when $c(xv)$ is zero for all or all but one $v \in V - \{x\}$. In the latter case updating $c(xv) := 0$ obviously preserves $\tau(c)$ and remains c inner Eulerian. Thus, upon termination of the iteration we can remove the node x from the set V.

The first stage finishes when the current V is just T. For the resulting c the optimal multiflow f is obvious. The aim of the second stage is to restore the desired optimal solution for the initial V and c. This is done in a natural way, by treating the nodes x and pairs $\{y, z\}$ in the order reverse to that occurred in the first stage.

It remains to explain how to find the number α efficiently. First we examine α that equals the number b as above. For the resulting c' compute $\tau^*(c') = \tau(c')$ by solving linear program (1.4). If $\tau(c') = \tau(c)$, we are done. Otherwise, the argument in Section 2 shows the existence of a metric $m \in \mathcal{E}$ such that $m(xy) + m(xz) - m(yz) \in \{2, 4\}$ and $c' \cdot m = \tau(c') < \tau(c)$. Let $\varepsilon = \tau(c) - \tau(c')$. We now examine α to be $b_1 = b - \lfloor \varepsilon/4 \rfloor$ (where $\lfloor a \rfloor$ is the greatest integer not exceeding a). Compute $\tau(c'')$ for the resulting c''. One can see that if $\tau(c'')$ equals $\tau(c)$, then $\alpha = b_1$ is as required, and if not, then for any metric $m \in \mathcal{E}$ with $c'' \cdot m = \tau(c'')$ the only case $m(xy) + m(xz) - m(yz) = 2$ is possible. This implies that in the latter case the desired α is $b_1 - \varepsilon/2$, where $\varepsilon = \tau(c) - \tau(c'')$.

Hence, for each $\{y, z\}$ that we handle at a step of an iteration, computing the above number α is reduced to solving (1.4) at most twice. Since (1.4) is solvable in strongly polynomial time and the total number of steps throughout the algorithm is $O(|V|^3)$, the algorithm runs in strongly polynomial time.

Remark. The above algorithm is not "combinatorial" because it uses the ellipsoid method. For $H = K_{2,r}$ "purely combinatorial" algorithms to solve (1.2) and (1.7) (with f integral in the inner Eulerian case) can also be constructed but they run in pseudo-polynomial or weakly polynomial time (we omit these algorithms here). No "purely combinatorial" strongly polynomial algorithm for the problem in question is known at present.

4. Minimizable graphs

As mentioned in the Introduction, the minimum 0-extension problem (1.2) can be efficiently solved for each minimizable graph, in particular, for K_2 and $K_{2,r}$. Can we present other examples of such graphs? In this section we show how to construct a new minimizable graph, once we are given an arbitrary pair of minimizable graphs.

First of all we observe that the property of being minimizable can be stated in polyhedral terms (in (4.1) below). Given a connected graph $H = (T, U)$ and a set $V \supseteq T$, let $\mathcal{P} = \mathcal{P}_{H,V}$ be the set of extensions of H to V. Since $\mathcal{P}$ is described via linear constraints (cf. (1.4)), $\mathcal{P}$ is a polyhedron.

Consider the *dominant polyhedron*

$$\mathcal{D} = \mathcal{D}_{H,V} := \{x \in \mathbb{R}^{E_V} \ : \ x \geq m \ \text{ some } \ m \in \mathcal{P}\}$$

of $\mathcal{P}$. A metric m on V that is a vertex of $\mathcal{D}$ is called *H-primitive.* In other words, m is H-primitive if and only if there are no $m', m'' \in \mathcal{P}$ different from m such that $m \geq \lambda m' + (1-\lambda)m''$ for some $0 \leq \lambda \leq 1$. The minimizability of H is characterizad as follows:

(4.1) H is minimizable if and only if, for any $V \supseteq T$, an H-primitive metric on V is a 0-extension of H to V, and vice versa.

Indeed, it is easy to check that each 0-extension of H is H-primitive. By linear programming arguments, $m \in \mathcal{P}$ is a vertex of $\mathcal{D}$ if and only if there exists $c : E_V \to \mathbb{Z}_+$ such that $c \cdot m < c \cdot m'$ for any other vector m' in $\mathcal{P}$. Now (4.1) follows from the fact that the nonnegative integer vectors c on E_V one-to-one correspond to the graphs G on V (with $c = c^G$).

The next lemma suggests a way to construct new minimizable graphs.

Lemma 4.1. *Let T' and T'' be subsets of T such that $T' \cup T'' = T$ and $T' \cap T''$ consists of a single element s. Let $H = (T, U)$ be the union of graphs $H' = (T', U')$ and $H'' = (T'', U'')$. Let both H' and H'' be minimizable. Then H is minimizable as well.*

Proof. Obviously d^H coincides with $d^{H'}$ and $d^{H''}$ within T' and T'', respectively. Consider an H-primitive metric m on a set $V \supseteq T$. Let V' be the set of $x \in V$ such that

$$m(sx) + m(xp) = m(sp) \quad (= d^H(sp)) \tag{4.2}$$

for some $p \in T'$, and let $V'' = (V - V') \cup \{s\}$. It is easy to see that $V' \cap T = T'$. This implies $V'' \cap T = T''$. First we assert that

(4.3) for any $x \in V'$ and $y \in V''$, $m(xy) = m(sx) + m(sy)$.

Indeed, this is trivial if some of $m(sx)$ and $m(sy)$ is zero, so assume that $m(sx)$, $m(sy) > 0$. We observe that there exist $u, q \in T$ such that $m(us) + m(sy) + m(yq) = m(uq)$. For otherwise one can decrease m on sy and, possibly, some other edges preserving the nonnegativity, triangle inequalities and constraints $m(tt') = d^H(tt')$ for all $t, t' \in T$, thus coming to a contradiction with the H-primitivity of m. The above equality implies

$$m(sy) + m(yq) = m(sq). \tag{4.4}$$

Since $y \in V''$, (4.4) shows that $q \in T''$. Choose $p \in T'$ satisfying (4.2) for our x. Since s is an only common node in H' and H'', we have $m(ps)+m(sq) = d^H(ps)+d^H(sq) = d^H(pq) = m(pq)$. This together with (4.2) and (4.4) yields (4.3), as required.

Let m' and m'' be the restrictions of m to V' and V'', respectively. We observe that m' is H'-primitive. For otherwise there are $m'_1, m'_2 \in \mathcal{P}_{H',V'}$ different from m' such that $m' \geq \lambda m'_1 + (1-\lambda)m'_2$ for some $0 \leq \lambda \leq 1$. For $i = 1, 2$ define $m_i(uv)$ to be $m'_i(uv)$ for $u, v \in V'$, $m(uv)$ for $u, v \in V''$, and $m'_i(us) + m(sv)$ for $u \in V'$ and $v \in V''$. One can see that both m_1 and m_2 are metrics in $\mathcal{P}_{V,H}$, and now, using (4.3), we conclude that $m \geq \lambda m_1 + (1-\lambda)m_2$, contrary to the H-primitivity of m.

Since H' is minimizable, m' is a 0-extension of H' to V' (by (4.1)). Similarly, m'' is a 0-extension of H'' to V''. These facts and (4.3) imply that m is a 0-extension of H to V, and now the lemma follows from (4.1). •

Repeatedly applying Lemma 4.1 to copies of the minimizable graph K_2, we obtain the following.

Corollary 4.2. *Every tree is minimizable.* •

Remark. One can generalize the concept of minimizability considering arbitrary metrics μ on T. An extension and 0-extension of μ and the numbers $\tau = \tau(G, \mu)$ and $\tau^* = \tau^*(G, \mu)$ are defined in an obvious way (replacing d^H by μ in (1.2) and (1.3)), and we say that μ is minimizable if τ and τ^* coincide for each $G = (V, E)$ with $V \supseteq T$. Lemma 4.1 is also extended to this general case. More precisely, given metrics μ' on T' and μ'' on T'', define $\mu(uv)$ to be $\mu'(uv)$ for $u, v \in T'$, $\mu''(uv)$ for $u, v \in T''$, and $\mu(us) + \mu(sv)$ for $u \in T'$ and $v \in T''$. Then μ is minimizable if both μ' and μ'' are so.

Recently the first author found a complete characterization (in combinatorial terms) of the set of minimizable graphs [5].

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