SUMS OF CUTS AND BIPARTITE METRICS

Alexander V. Karzanov ♯

Abstract. Let m be an integer-valued metric on a finite set V such that the length of any circuit on V is even, and let $H = (VH, EH)$ be an undirected graph with $VH \subseteq V$. A family $\{m_1, \dots, m_k\}$ of metrics on V is called an H-packing for m if the value $m_1(x, y) + \dots + m_k(x, y)$ does not exceed $m(x, y)$ for any $x, y \in V$ and equals $m(x, y)$ for each edge $xy \in EH$. A metric m' on V is said to be induced by a graph G if there is a mapping σ from V onto VG such that for any $x, y \in V$, $m'(x, y)$ is equal to the distance in G from $\sigma(x)$ to $\sigma(y)$. It is known that if $|VH| \leq 4$ then there exists an H-packing for m consisting of metrics induced by the graph K_2 (i.e., cut metrics), and this is, in general, false when $|VH| > 4$.

We prove that if $|VH| = 5$ then there exists an H-packing for m consisting of metrics induced either by the graph K_2 or by the graph $K_{2,3}$. Also other results on packings and decompositions of metrics are presented.

Keywords. Finite metric, cut cone, multicommodity flow.

1. Introduction

Throughout the paper, by a *graph* we mean a finite undirected graph without loops and multiple edges; VG is the vertex set and EG is the edge set of a graph G. An edge with end vertices x and y may be denoted by xy.

Let V be a finite set of n elements, and (V, m) a semimetric space, i.e., $m(x, x) = 0$, $m(x, y) = m(y, x) \geq 0$ and m satisfies the triangle inequalities $m(x, y) + m(y, z) \geq m(x, z)$, $x, y, z \in V$. For brevity, we refer to m as a *metric* (rather than a semimetric) *on* V; $m(x, y)$ will be denoted by $m(xy)$. A metric m is called *positive* if $m(xy) > 0$ for all distinct $x, y \in V$.

We may identify m with the corresponding function on the edge set EK_V of the complete graph K_V with the vertex set V. Then the set of metrics on V forms a polyhedral cone $\mathcal{M}_V$ in the $\binom{n}{2}$-dimensional euclidean space $\mathbb{R}^{EK_V}$ (whose coordinates correspond to the edges of K_V), called the *metrical cone*. The metrics belonging to extreme rays in $\mathcal{M}_V$ are called *primitive*. The obvious fact is that if $m(xy) = 0$ for some $x \neq y$, then m is primitive if and only if the corresponding metric on the set obtained from V by identifying x and y is primitive.

For a connected graph G, we say that a metric m' on V is *induced by* G if there is a mapping σ from V *onto* VG such that $m'(xy) = d^G(\sigma(x)\sigma(y))$ for $x, y \in V$, where $d^G(uv)$ denote the distance in G between vertices u and v (assuming that the length of each edge of G is 1). An elementary example gives a metric m' induced by the graph

♯ Institute for System Studies, Prospect 60 let Oktyabrya, 9, 117312 Moscow, USSR

K_2 (where K_p is the complete graph with p vertices). Such an m' is often called a *cut metric*. In other words, $m' = \rho X$ is generated by a proper subset X of V as follows:

$$\begin{aligned} \rho X(xy) &:= 1 \quad \text{if } |\{x,y\} \cap X| = 1, \\ &:= 0 \quad \text{otherwise.} \end{aligned} \tag{1}$$

Note that m' is primitive since the metric d^{K_2} is obviously primitive. For a cut metric m' and a real number $t \geq 0$, the metric tm' is called a *Hamming* metric.

One natural problem on metrics is:

(D): *given a metric m on V, decide whether m is decomposable into a sum*

$$m = m_1 + \ldots + m_k, \tag{2}$$

where $m_1, \ldots, m_k$ are metrics from a certain collection S.

In particular, if S is the set S_1 of Hamming metrics, we obtain the membership problem: decide whether a metric m is contained in the *Hamming* (or *cut*) *cone* $\mathcal{H}_V$. [$\mathcal{H}_V$ is the convex hull of the set of Hamming metrics on V. An equivalent definition: $\mathcal{H}_V$ is the set of metrics on V such that (V, m) is embeddable isometrically into L^1 (see [1]).] Unfortunately, already for this "simplest" collection S_1 the problem (D) turns out to be NP-hard; this follows from the NP-hardness of the separation problem for $\mathcal{H}_V$ [7] and the fact that the membership problem and the separation problem are polynomially equivalent for a large class of convex sets [6]. It was proved in [3] that the problem is NP-complete if S is the set S_2 of cut metrics. However, a number of nontrivial sufficient conditions on a metric to be decomposable into a sum of metrics in S_1 or S_2 is known. See, e.g., [4,5]. One more of them will be pointed out in statement (1.4) below.

Now we introduce a notion closely related to decompositions of metrics. Let G be a connected graph with $VG = V$ whose edges $e \in EG$ have nonnegative real-valued *lengths* $l(e)$, and let H be a graph with $VH \subseteq V$.

Definition. A family $\{m_1, \ldots, m_k\}$ of (possibly repeated) metrics on V is called an *H-packing* for l if

$$l(xy) \geq m_1(xy) + \ldots + m_k(xy) \quad \text{for all } \; x, y \in V; \tag{3}$$

and

$$d_l(s,t) = m_1(st) + \ldots + m_k(st) \quad \text{for all } \; st \in EH. \tag{4}$$

Here $d_l(xy)$ denotes the distance between vertices $x, y \in VG$ with respect to l. In order to demonstrate a relation between packings and decompositions we need the following definition.

Definition. An *extremal graph* of a metric m on V is a minimal (with respect to inclusion) graph H such that for any $u, v \in V$, there is an edge $st \in EH$ satisfying

$$m(su) + m(uv) + m(vt) = m(st). \tag{5}$$

[In [10,11] the term "antipode graph" was introduced for such a graph in the case of a positive metric.] For example, if m is a cut metric ρX, then a graph H is extremal for m if and only if it consists of two vertices $x \in X$ and $y \in V - X$ and one edge xy.

Considering (2)–(5) for $G = K_V$ and $l = m$ and applying triangle inequalities we easily obtain the following.

(1.1) *If H is an extremal graph for a metric m and $\mathcal{F} = \{m_1, \ldots, m_k\}$ is an H-packing for m, then $\mathcal{F}$ is a decomposition of m, i.e., (2) is valid.* •

The problem which will be in focus of the present paper is: for any fixed H, determine a minimal collection S of metrics so that, for any connected graph G with $VG \supseteq VH$ and any length function l on EG, there exists an H-packing for l consisting of metrics in S. Such a problem is related to multicommodity flows, as we explain in Section 4. In particular, if S is the collection of Hamming metrics, such a relation enables us to derive the following statement from a multicommodity flow theorem of Papernov [12]:

(1.2) *If H is K_4 or C_5 or a union of two stars, then there exists an H-packing for l consisting of Hamming metrics.* •

[A *star* is a connected graph whose edges have a common vertex; C_5 is the circuit with five vertices. It is easy to see that any proper subgraph of K_4 or C_5 having no isolated vertices is a union of two stars.] Statement (1.2) cannot be strengthened in term of the graphs H; more precise, one can prove that if H is not as in (1.2) and it contains no isolated vertex, then for any $V \supseteq VH$ there exists a metric m on V having no H-packing of Hamming metrics. There is a stronger, "half-integral", version of (1.2). We say that a (nonnegative) function l is *cyclically even* if it is integer-valued and each circuit in G has an even length, i.e., $l(x_0x_1) + \ldots + l(x_{r-1}x_r) + l(x_rx_0)$ is even for any $x_0x_1, \ldots, x_{r-1}x_r, x_rx_0 \in EG$.

(1.3) [7] *If H is as in (1.2) and l is cyclically even, then there exists an H-packing for l consisting of cut metrics.* •

[Another, simpler, proof of (1.3) is given in [13].] Assertions (1.1) and (1.3) imply the following:

(1.4) *If a metric m has K_4 or C_5 or a union of two stars as an extremal graph, then $m \in \mathcal{H}_V$. If, in addition, m is cyclically even, then m is decomposable into a sum of cut metrics.* •

A simplest example of a metric not decomposable into a sum of cut metrics gives any metric $d^{K_{p,q}}$ for $p \geq 2$ and $q \geq 3$ ($K_{p,q}$ is the complete bipartite graph with parts of p and q vertices). It is known that such a metric is primitive; see, for example, [10,2]. The metric induced by the graph $K_{p,q}$ is called *p, q-metric.*

The main result of the present paper is the following.

Theorem 1. *If l is a cyclically even function on the edges of a connected graph G, H is a graph with $VH \subseteq VG$ and $|VH| = 5$, then there exists an H-packing for l consisting of cut metrics and 2,3-metrics.*

(1.5) (Corollary from (1.1) and Theorem 1) *Every cyclically even metric having an extremal graph H with $|VH| = 5$ is representable as a sum of cut metrics and 2,3-metrics.* •

Theorem 1 will be proved in Section 2; the proof will provide a strongly polynomial algorithm for finding a required packing. Note that, in fact, a slightly stronger version of this theorem will be proved in which one asserts that all used 2,3-metrics can be chosen to coincide on the set VH. In particular, if m is a cyclically even metric on a set V of five elements, then m is a sum of cut metrics and of some number of copies of one 2,3-metric on V.

Theorem 1 can be reformulated in polyhedral terms as follows (a corresponding statement can be stated also for (1.3)). For $U \subset EK_V$, define the cone $\mathcal{M}_{V,U}$ to be the nonnegative linear hull of the cone $\mathcal{M}_V$ and the vectors I_e, $e \in EK_V - U$, where I_e is the e-th unit basis vector in $\mathbb{R}^{EK_V}$ (such a cone occurred in [10,11]).

(1.6) *Let the edges in U span exactly 5 vertices. Then:*

(i) *each extreme ray of $\mathcal{M}_{V,U}$ is $\{\lambda a : \lambda \geq 0\}$, where a is either I_e $(e \in EK_V - U)$ or a cut metric or a 2,3-metric;*

(ii) *if l is an integral vector in $\mathcal{M}_{V,U}$ such that $l(xy) + l(yz) + l(xz)$ is even for any distinct $x, y, z \in V$, then l is a sum of integral vectors lying on extreme rays in $\mathcal{M}_{V,U}$.*

Section 3 contains some generalizations. By the *metrical spectrum* $MS(H)$ of a graph H we will mean the minimal set S so that: (i) each $d \in S$ is an integer-valued metric on a set $V(d)$, and td is not integer-valued for $0 < t < 1$; and (ii) for any connected graph G with $VG \supseteq VH$ and a function $l : EG \to \mathbb{R}_+$, there exist metrics $d_1, \ldots, d_k \in S$ and reals $\lambda_1, \ldots, \lambda_k \geq 0$ such that $\{\lambda_1 m_1, \ldots, \lambda_k m_k\}$ is an H-packing for l, where m_i is a metric on VG induced by d_i, i.e., $m_i(xy) = d_i(\sigma(x)\sigma(y))$, $x, y \in V$, for some mapping σ from VG onto $V(d_i)$. For example, the metrical spectrum of K_4 consists uniquely of the metric d^{K_2} (by (1.2)), and the metrical spectrum of K_5 consists of the two metrics d^{K_2} and $d^{K_{2,3}}$ (by Theorem 1). We give in Section 3 a complete description of the set of graphs H for which $MS(H)$ is finite.

2. Proof of Theorem 1

Put $T := VH$ and $E := EK_V$. It suffices to prove that if $|T| \leq 5$ and m is a cyclically even function (not necessarily a metric) on E, then there exists a K_T-packing for m consisting of cut metrics and 2,3-metrics. Our method of proof uses ideas of [8], developed there for proving that if m' is a primitive metric having an extremal graph

H' with $|VH'| = 5$, then m' is proportional to 2,3-metric. In particular, the statements (2.1),(2.3) and (2.4) occurred in [8]; we give here their proofs in order to make our description self-contained.

By a *path*, or $x - y$ *path*, on V we mean a sequence $P = x_0 x_1 \dots x_k$ of distinct elements $x = x_0, x_1, \dots, x_k = y$ of V. $e_i = x_i x_{i+1}$ is an *edge* of P and the value $m(P) = \sum(m(e_i) : i = 1, \dots, k)$ is the *length* of P (with respect to m); P is *shortest* if $m(P) = d_m(xy)$.

For $xy \in E$ and $u, v, w \in V$ put:

$$\phi(xy) = \phi_m(xy) := \min\{d_m(sx) + d_m(xy) + d_m(yt) - d_m(st) : s, t \in T\};$$

$$\Delta(u, v, w) = \Delta_m(u, v, w) := d_m(uv) + d_m(vw) - d_m(uw).$$

The cyclically evenness of m implies that $\phi(xy)$ and $\Delta(u, v, w)$ are even.

We proceed by induction on

$$\begin{aligned} \alpha = \alpha(V, T, m) :=& |V| + |\{e \in E : m(e) > 0\}| + |\{e \in E : \phi(e) > 0\}| \\ &+ |\{(s, x, t) : x \in V,\ s, t \in T,\ \Delta(s, x, t) > 0\}|. \end{aligned}$$

By (1.3), the theorem is true if $|T| \le 4$.

First of all we show that one may consider only the case when the following hold:

$$m \text{ is a positive metric;} \tag{6}$$

$$\text{each } e \in E \text{ is contained in some shortest } s - t \text{ path, } s, t \in T, \text{ i.e. } \phi(e) = 0; \tag{7}$$

$$\text{each } p \in T \text{ is contained in some shortest } s - t \text{ path with } s, t \in T - \{p\}. \tag{8}$$

This is achieved by use of the following simple reductions.

(i) Suppose that $m(xy) = 0$ for some $xy \in E$. Identify x and y with a new vertex z, obtaining corresponding V', T'. For $u, v \in V'$, define $m'(uv) := m(uv)$ if $u, v \ne z$, and $m'(uv) := \min\{m(ux), m(uy)\}$ if $v = z$. Then m' is cyclically even and $\alpha(V', T', m') < \alpha(V, T, m)$. By induction there exists a required $K_{T'}$-packing for m', which naturally determines a required K_T-packing for m.

Thus, one can assume that $m(e) > 0$ for all $e \in E$.

(ii) Suppose that $m(e) \ge 2$ and $\phi(e) > 0$ for some $e \in E$. Let a be the maximum even number not exceeding $\max\{m(e), \phi(e)\}$. Put $m'(e) := m(e) - a$ and $m'(e') := m(e')$, $e' \in E - \{e\}$. Clearly, m' is cyclically even, $d_{m'}(st) = d_m(st)$ for all $s, t \in T$, and $\alpha(V, T, m') \le \alpha(V, T, m)$. Three case are possible: (a) $m'(e) = 0$, (b) $\phi_{m'}(e) = 0$, and (c) $m'(e) = 1$. In cases (a) and (b), we have $\alpha(V, T, m') < \alpha(V, T, m)$, and the result follows by induction. So we may assume that $\phi(e) = 0$ for all $e \in E$ with $m(e) \ge 2$. Suppose there is an edge $xy \in E$ with $m(xy) = 1$. Take $z \in V - \{x, y\}$, and let for definiteness $m(xz) \ge m(yz)$. Since $m(xy) + m(xz) + m(yz)$ is even and m is positive, we have $m(xz) \geq 2$ and $m(xy) + m(yz) = m(xz)$. Since $m(xz) \geq 2$, $\phi(xz) = 0$, and hence there exists a shortest path $s \dots xz \dots t$ with $s, t \in T$. Then the path $s \dots xyz \dots t$ is also shortest.

Thus, one can assume that (7) holds. It follows easily from (7) that m is a metric, whence (6) holds.

(iii) Suppose that $\omega(p) > 0$ for some $p \in T$, where $\omega(p) := \min\{\Delta(s,p,t) : s,t \in T - \{p\}\}$. Put $X := \{p\}$ and $m' := m - a\rho X$, where

$$a := \min\{\omega(p)/2, \min\{m(py) : y \in V - \{p\}\}\}.$$

Since a is integer, m' is cyclically even. Obviously, $d_{m'}(st) = d_m(st)$ and $d_{m'}(sp) = d_m(sp) - a$ for $s,t \in T - \{p\}$. Furthermore, at least one of the following is true: $\Delta_{m'}(s,p,t) = 0$ for some $s,t \in T - \{p\}$, or $m'(xy) = 0$ for some $y \in V - \{p\}$, whence $\alpha(V,T,m') < \alpha(V,T,m)$. By induction there exists a required packing for m'. Adding to it a copies of the cut metric ρX, we obtain a required packing for m.

Thus, one can assume that (8) holds.

Let H' be the extremal graph for m (it is easy to show that a positive metric has a unique extremal graph). Let $U := EH'$. (7) and (8) imply

$$\text{each edge } e \in E \text{ is in some shortest } s-t \text{ path for } st \in U; \tag{9}$$

$$\text{for each } p \in T, \text{ there is } st \in U \text{ such that } s,t \neq p \text{ and } m(sp) + m(pt) = m(st). \tag{10}$$

In view of (1.3), one can assume that H' is different from K_4, C_5 and a union of two stars. In particular, this implies that $|VH'| = 5$, i.e., $VH' = T$. Also one can show that there are three vertices in H', say, s_1, s_2, s_3, such that $s_i s_j \in U$, $1 \leq i < j \leq 3$ (otherwise H' is either C_5 or a union of two stars). Let $T_1 := \{s_1, s_2, s_3\}$ and $T_2 := T - T_1 =: \{s_4, s_5\}$.

(2.1) (i) $U = \{s_1s_2, s_2s_3, s_3s_1, s_4s_5\}$;
(ii) $m(s_is_4) + m(s_is_5) = m(s_4s_5)$ for $i = 1,2,3$.

Proof. Consider $p \in T_1$. Let s and t be vertices as in (10). The minimality of the extremal graph H' implies $ps, pt \notin U$. Thus, $\{s,t\} \cap T_1 = \emptyset$, i.e., $\{s,t\} = \{s_4, s_5\}$. Hence, $s_4s_5 \in U$, $ps_4, ps_5 \notin U$, and (ii) is true. •

For $\emptyset \neq X \subset V$, let δX denote the set with one end in X and the other in $V - X$ (a *cut* on V); δX is said to *separate* vertices x and y if $|\{x,y\} \cap X| = 1$. As before, for a metric m one assumes by definition that $m(xx) = 0$, $x \in V$. For $x, y \in V$, let $N(x,y)$ denote the set of vertices contained in shortest $x - y$ paths.

(2.2) *Let $s \in T_1$, $t \in T_2$, $\{p,q\} = T_1 - \{s\}$, and let $N(s,t) \cap N(p,q) = 0$. Put $X := N(s,t)$,*

$$a := \frac{1}{2}\min\{\min\{m(sz) + m(zt) - m(st) \ : \ z \in V - X\},$$
$$\min\{m(pz) + m(zq) - m(pq) \ : \ z \in X\}\},$$

and $m' := m - a\rho X$. Then $m' \ge 0$, a is an integer ≥ 1, and for each $uv \in U$ the following holds:

$$\begin{aligned} d_{m'}(uv) &= m(uv) - a \quad \text{if } \delta X \text{ separates } u \text{ and } v, \\ &= m(uv) \qquad\quad \text{otherwise.} \end{aligned} \tag{11}$$

Proof. The condition that $N(s,t)$ and $N(p,q)$ are disjoint and the cyclically evenness of m imply that a is an integer ≥ 1. Next, the vertex r in T_2 different from t cannot be in X, by the minimality of H'.

(i) Consider vertices $x, y \in X$, and let for definiteness $m(sx) \le m(sy)$. We assert that the path $sxyt$ is shortest (for m). This is true for $x = y$ by the definition of $N(s,t)$. Let $x \ne y$. By (9), there is a shortest path $s'xyt'$ for some $s't' \in U$. Since x belongs to no shortest $p-q$ path, we have $s't' \ne pq$. Therefore, $s't' \in \{sp, sq, tr\}$. Suppose $s't' = sp$. It follows from $m(s'x) \le m(s'y)$ that $s' = s$. Since the paths syt and $s'xy$ are shortest, the path $sxyt$ is also shortest, as required. The cases $s't' = sq, tr$ are considered analogously.

(ii) Let $x, y \in X$ and $z \in V - X$, and let for definiteness the path $sxyt$ be shortest. Then

$$\begin{aligned} m(xz) + m(zy) - m(xy) = m(sx) &+ m(xz) + m(zy) + m(yt) - m(st) \\ &\ge m(sz) + m(zt) - m(st) \ge 2a, \end{aligned} \tag{12}$$

by the definition of a. If $x = y$, we obtain from (12) that $m(xz) \ge a$, whence $m' \ge 0$.

(iii) Let $uv \in U$. Consider a $u-v$ path $P = x_0x_1\dots x_k$ shortest for m'. One must prove that $m'(P)$ is equal to $m(uv)$ if $uv = pq$ and equal to $m(uv) - a$ if $uv = sp, sq, tr$. One can assume that k is minimum (by uv fixed). The assertion is obvious when $k = 1$. Let $k \ge 2$. For $i < k-1$, let P_i denote the path $x_0x_1\dots x_ix_{i+2}\dots x_k$; then $m'(P_i) > m'(P)$. We observe that P has the following properties.

(a) $x_ix_{i+1} \in \delta X$. Indeed, suppose that $x_i, x_{i+1} \in X$, and let for definiteness $i \le k-2$. If $x_{i+2} \in X$, then $m'(e) = m(e)$ for $e = x_ix_{i+1}, x_ix_{i+2}, x_{i+1}x_{i+2}$, and if $x_{i+2} \in V - X$, then $m'(x_ix_{i+1}) = m(x_ix_{i+1})$ and $m'(e) = m(e) - a$ for $e = x_ix_{i+2}, x_{i+1}x_{i+2}$. In both cases, we obtain from $m(x_ix_{i+1}) + m(x_{i+1}x_{i+2}) \ge m(x_ix_{i+2})$ that $m'(P_i) \le m'(P)$; a contradiction with the minimality of k. The case $x_i, x_{i+1} \in V - X$ is considered similarly.

(b) There is no i such that $x_i, x_{i+2} \in X$ and $x_{i+1} \in V - X$. Otherwise, taking into account (12), we have

$$\begin{aligned} m'(x_ix_{i+2}) = m(x_ix_{i+2}) \le m(x_ix_{i+1}) &+ m(x_{i+1}x_{i+2}) - 2a \\ &= m'(x_ix_{i+1}) + m'(x_{i+1}x_{i+2}), \end{aligned}$$

whence $m'(P_i) \le m'(P)$; a contradiction.

It follows from (a) and (b) that $k=2$, $x_0, x_2 \in V-X$ and $x_1 \in X$. Then $uv = pq$, and now we conclude from the definition of a that $m'(P) = m(px_1) + m(x_1q) - 2a \geq m(pq)$. $\bullet$

Suppose that s, t, a, X and m' are so as in (2.2). It follows easily from (11) that $\phi_{m'}(e) \leq \phi_m(e)$ and $\Delta_{m'}(s', x, t') \leq \Delta_m(s', x, t')$ for all $e \in E$, $x \in V$ and $s', t' \in T$. Furthermore, m' is cyclically even (as a is an integer) and $\Delta_{m'}(s', z, t') = 0$ for $s't' = st$ and some $z \in V - X$ or for $s't' = pq$ and some $z \in X$. Therefore, $\alpha(V, T, m') < \alpha(V, T, m)$, and by induction there exists a required packing for m'. Adding to it a copies of the cut metric ρX we obtain a required packing for m.

So we may assume that $N(p, q) \cap N(s, t) \neq \emptyset$ whenever $\{p, q, s\} = T_1$ and $t \in T_2$. Put $a_{ij} := m(s_i s_j)$ for $i = 1, 2, 3$, $j = 4, 5$; $b_{ij} = b_{ji} := m(s_i s_j)$ for $1 \leq i < j \leq 3$; and $c := m(s_4 s_5)$.

(2.3) *Let $\lambda := a_{14}$. Then all a_{ij} are equal to λ and $b_{12} = b_{23} = b_{31} = c = 2\lambda$.*

Proof. Let $i \in \{1, 2, 3\}$ and $j \in \{4, 5\}$. Choose a vertex x in $N(s_i, s_j)$ contained in a shortest $s_k - s_r$ path for $\{i, k, r\} = \{1, 2, 3\}$. Then

$$a_{ij} + b_{kr} = m(s_i x) + m(s_j x) + m(s_k x) + m(s_r x) \geq a_{kj} + b_{ir}.$$

Therefore, $a_{1j} + b_{23} = a_{2j} + b_{13} = a_{3j} + b_{12}$. Considering these equalities for $j = 4, 5$ and the equalities in (2.1)(ii) we obtain $a_{1j} = a_{2j} = a_{3j} =: a_j$ and $b_{12} = b_{23} = b_{31} =: b$. Now since s_j is in a shortest $s_i - s_k$ path for some $1 \leq i < j \leq 3$ (by (10)), we have $2a_j = b$, whence $a_1 = a_2 = \lambda$ and $b = c = 2\lambda$. $\bullet$

Assertion (2.3) shows that the restriction of m on T is a metric proportional to $d^{K_{2,3}}$. Now our aim is to show that for a metric with such a property there exists a packing consisting of 2,3-metrics.

For $i, j = 1, \ldots, 5$, put $N_{ij} := N(s_i, s_j)$. Define the sets:

$$\begin{aligned} &S_4 := \{s_4\}; \\ &S_i := N_{i4} - \{s_4\}, \quad i = 1, 2, 3; \\ &S_5 := V - (S_1 \cup S_2 \cup S_3 \cup S_4). \end{aligned}$$

Clearly $s_i \in S_i$, $i = 1, 2, 3, 4$. Also (2.3) implies that $s_5 \in S_5$. Below we shall prove the following:

(2.4) *The sets $S_1, \ldots, S_5$ are disjoint.*

The partition $\mathcal{P} := \{S_1, \ldots, S_5\}$ of V defines the 2,3-metric d on V as

$$\begin{aligned} d(xy) &:= 1 \quad \text{if } xy \in (S_i, S_j),\ i = 1, 2, 3,\ j = 4, 5; \\ &:= 2 \quad \text{if } xy \in (S_1, S_2), (S_2, S_3), (S_3, S_1), (S_4, S_5); \\ &:= 0 \quad \text{otherwise,} \end{aligned}$$

where (X, Y) is the set of edges with one end in X and the other in Y. By (2.3), $m(st) = \lambda d(st)$, $s, t \in T$. Put

$$\beta := \min\{m(s_4x) \ : \ x \in S_1 \cup S_2 \cup S_3\};$$
$$\gamma := \frac{1}{2}\min\{m(s_4x) + m(xs_i) - m(s_4s_i) \ : \ x \in S_5, \ i = 1, 2, 3\},$$

and $a := \min\{\beta, \gamma\}$. Then a is an integer ≥ 1, as follows directly from the definition of $S_1, \ldots, S_k$. Put $m' := m - ad$. Below we shall prove the following:

(2.5) *$m' \geq 0$ and $d_{m'}(st) = m(st) - ad(st)$ for all $st \in U$.*

In the assumption that (2.4) and (2.5) are true, the proof of Theorem 1 is completed as follows. It is easy to check that the metric $d^{K_{2,3}}$ is cyclically even, whence, in view of the integrality of a, the metric m' is cyclically even. It follows from the definition of a that $\alpha(V, T, m') < \alpha(V, T, m)$. By induction there exists a required packing for m'. Adding to it a copies of the metric d yields a required packing for m.

Proof of (2.4). It suffices to prove that $N_{i4} \cap N_{j4} = \{s_4\}$ for $1 \leq i < j \leq 3$. Suppose that $N_{i4} \cap N_{j4}$ contains a vertex x different from s_4. Since $m(s_4x) > 0$ and $m(s_ix) + m(xs_4) = \lambda$ (where λ is defined as in (2.3)), then $m(s_ix) < \lambda$; similarly $m(s_jx) < \lambda$. Hence, $m(s_is_j) < 2\lambda$, contrary to (2.3). •

Proof of (2.5). First of all we make several preliminary observations.

$$m'(xy) = m'(xs_4) + m'(s_4y) \quad \text{for } x \in S_i,\ y \in S_j,\ 1 \leq i < j \leq 3; \tag{13}$$

$$m'(xz) + m'(zy) \geq m'(xs_4) + m'(s_4y) \ \text{ for } x \in S_i,\ y \in S_j,\ 1 \leq i < j \leq 3,\ z \in S_5. \tag{14}$$

Indeed, since $m(s_ix) + m(xs_4) = \lambda = m(s_jy) + m(ys_4)$ and $m(s_is_j) = 2\lambda$, the path $s_ixs_4ys_j$ is shortest for m, whence $m(xy) = m(xs_4) + m(s_4y)$ and $m(xz) + m(zy) \geq m(xs_4) + m(s_4y)$. Now (13) and (14) follow from the fact that the value $m(e) - m'(e)$ is a for $e \in (S_k, S_r)$, $k = 1, 2, 3$, $r = 4, 5$, and $2a$ for $e \in (S_i, S_j)$.

$$m'(xz) + m'(zy) \geq m'(xy) \quad \text{for } x \in S_i \cup S_4,\ y \in S_i,\ 1 \leq i \leq 3, \text{ and } z \in S_5 \tag{15}$$

(where if $x = y$ then $m'(xy) := 0$). Indeed, as it was shown earlier (see (12)), $\Delta(x, z, y) \geq \Delta(s_4, z, s_i)$, where $\Delta(x', z', y')$ is $m(x'z') + m(z'y') - m(x'y')$. Thus, $\Delta(s_4, z, s_i) \geq 2a$ (by the definition of a) implies $\Delta(x, z, y) \geq 2a$. If $x \in S_i$ then $m'(xy) = m(xy)$ and $m'(e) = m(e) - a$ for $e = xz, xy$, and if $x = s_4$ then $m'(xz) = m(xz) - 2a$ and $m'(e) = m(e) - a$ for $e = xy, zy$, whence (15) follows.

$$m'(s_4x) \geq 0 \quad \text{for } x \in S_5. \tag{16}$$

Indeed, let for definiteness $m(s_1x) \leq m(s_2x) \leq m(s_3x)$. Suppose that $m(s_1x) + m(s_2x) > 2\lambda$. Then $m(s_2x) > \lambda$ and the edge s_2x belongs to no shortest $s - t$

paths for $st = s_1s_2, s_2s_3, s_3s_1$. Hence, s_2x is in a shortest $s_4 - s_5$ path (by (9)). But $m(s_4s_2) = m(s_2s_5) = \lambda$; a contradiction. Thus, $m(s_1x) + m(s_2x) = 2\lambda$. Now we get from $\Delta(s_4, x, s_i) \geq 2a$, $i = 1, 2$, and $m(s_1s_4) + m(s_4s_2) = 2\lambda$ that $m(s_4x) \geq 2a$, whence (16) follows.

We show $m'(uv) \geq 0$ for all u, v. Let $u \in S_i$ and $v \in S_j$. If $i = j$ then $m'(uv) = m(uv) \geq 0$. If $i \neq j$, then $m'(uv) \geq 0$ follows for $i = 4$, $j = 5$ from (16), and for $i = 4$, $j = 1, 2, 3$ from the definition of a. Therefore, $m'(uv) \geq 0$ for $\{i, j\} = \{1, 2\}, \{2, 3\}, \{3, 1\}$, by (13). Finally, this follows for $i = 1, 2, 3$, $j = 5$ from (15) (putting $x = y = u$ and $z = v$).

Finally we show the second half of (2.5). Let $st \in U$ be fixed. Consider an $s - t$ path $P = x_0x_1 \ldots x_k$ shortest for m'. One must prove that $m'(P) = m(st) - ad(st)$. We may assume that P satisfies the following conditions: (i) the number of vertices x_i different from s_4 is minimum; and (ii) the number of indices i such that $x_i = s_4$ is maximum subject to (i). Let P_i denote the path $x_0x_1 \ldots x_ix_{i+2} \ldots x_k$. We observe that P satisfies the following four properties.

(a) x_i and x_{i+1} belong to different sets in $\mathcal{P}$. Indeed, if, say, $x_i, x_{i+1} \in S_k$, $x_{i+2} \in S_r$, $k \neq r$, then $x_{i+1} \neq s_4$ and we have from $m'(x_ix_{i+1}) = m(x_ix_{i+1})$ and $m'(e) = m(e) - ad(s_ks_r)$ for $e = x_{i+1}x_{i+2}, x_ix_{i+2}$ that $m'(P_i) \leq m'(P)$, which contradicts (i).

(b) There is no i such that $x_i \in S_k$ and $x_{i+1} \in S_r$ for $k, r \in \{1, 2, 3\}$, $k \neq r$. Otherwise the path $x_0x_1 \ldots x_is_4x_{i+1} \ldots x_k$ is shortest for m' (by (13)), contrary to (ii).

(c) There is no i such that $x_i \in S_k \cup S_4$, $x_{i+1} \in S_5$ and $x_{i+2} \in S_r \cup S_4$ for $k, r \in \{1, 2, 3\}$. Otherwise $m'(P_i) \leq m'(P)$ (by (14) or (15)), contrary to (i).

(d) There is no i such that $x_{i+1} \in S_k$ for $k \in \{1, 2, 3\}$ and either $x_i = s_4$ and $x_{i+2} \in S_5$, or $x_i \in S_5$ and $x_{i+2} = s_4$. Otherwise it follows from $m'(e) = m(e) - a$ for $e = x_ix_{i+1}, x_{i+1}x_{i+2}$ and from $m'(x_ix_{i+2}) = m(x_ix_{i+2}) - 2a$ that $m'(P_i) \leq m'(P)$, contrary to (i).

Now in the case $s = s_4$ and $t = s_5$, we conclude easily from (a)–(d) that $P = s_4s_5$. Then $m'(P) = m'(s_4s_5) = m(s_4s_5) - 2a$. In the case $s = s_k$ and $t = s_r$, $1 \leq k < r \leq 3$, we conclude from (a)–(d) that either $P = s_ks_r$ or $P = s_ks_4s_r$, whence $m'(P) = m(s_ks_r) - 2a$. •

This completes the proof of Theorem 1.

In fact, the proof of the theorem contains an algorithm for finding a required packing whose running time is a polynomial in $|V|$.

Remark. One can see that if m and m' are as in (2.5), then the metrics m and $d_{m'}$ are proportional on EK_T. Furthermore, obviously, $d_{m'}$ satisfies the properties as in (9) and (10). This gives the strengthening of Theorem 1 pointed out in the Introduction.

3. Metrical spectra of graphs

Theorem 1 has the following corollary.

(3.1) *Let H be a union of K_3 and a star, and let l be a cyclically even function on the edges of a connected graph G with $VG \supseteq VK$. Then there exists an H-packing for l consisting of cut metrics and 2,3-metrics.*

Proof. Let for definiteness H have the edges s_1s_2, s_2s_3, s_3s_1 and st_i, $i = 1, \ldots, r$ (possibly vertices in $\{s, t_1, \ldots, t_r\}$ coincide with some vertices in $\{s_1, s_2, s_3\}$). Add to G a new vertex t and the edges $t_1t, \ldots, t_rt$, forming the graph G'. Let l' be a cyclically even function on EG' such that $l'(e) = l(e)$ for $e \in EG$ and

$$d_{l'}(pq) = d_l(pq) \quad \text{for all } pq \in EH; \tag{17}$$

$$d_{l'}(st) = d_l(st_i) + l'(t_it) \quad \text{for } i = 1, \ldots, r; \tag{18}$$

it is easy to show that such a function exists. Let T be the set of different vertices among s_1, s_2, s_3, s, t; then $|T| \le 5$. By Theorem 1 there is a K_T-packing $\{m'_1, \ldots, m'_k\}$ for l' consisting of cut metrics and 2,3-metrics. Let m_j be the restriction of m'_j on VG. One can see that each m_j is a cut metric or a 2,3-metric or a sum of cut metrics. It obviously follows from (17) and (18) that $m_1, \ldots, m_k$ determine a required H-packing for l. •

Assertions (1.2),(3.1) and Theorem 1 give the metrical spectra $MS(H)$ when $|VH| \le 5$ or H is a union of two stars or H is a union of K_3 and a star; in these cases $MS(H)$ is either $\{d^{K_2}\}$ or $\{d^{K_2}, d^{K_{2,3}}\}$. Now we study the metrical spectra for the other graphs H. One can assume that H has no isolated vertices. A direct check-up shows that H is one of the following:

$$H \text{ has a matching of three edges;} \tag{19}$$

$$H \text{ consists of two disjoint graphs } K_3. \tag{20}$$

We assert that the set $MS(H)$ is infinite for any graph H as in (19), and that it is finite for the graph H as in (20).

Clearly if H'' is a subgraph of a graph H', then $MS(H'') \subseteq MS(H')$. Let H_0 be the graph consisting of three disjoint edges. So infiniteness of $MS(H)$ for H as in (19) is implied by the infiniteness of $MS(H_0)$. In order to show the infiniteness of $MS(H_0)$ take positive integers $p, q, r \ge 2$, and let G be the graph whose vertices are the triples (i, j, k), $i = 1, \ldots, p$, $j = 1, \ldots, q$, $k = 1, \ldots, r$, and whose edges are the pairs $\{(i, j, k), (i', j', k')\}$ such that either $|i - i'| + |j - j'| + |k - k'| = 1$ or $i - i' = j - j' = k - k' = 1$. Then the metric d^G is primitive and the edges of its extremal graph correspond to the pairs $\{(p, 1, 1), (1, q, r)\}, \{(1, q, 1), (p, 1, r)\}$ and $\{(1, 1, r), (p, q, 1)\}$ (see [8]). So d^G belongs to $MS(H_0)$. Hence $MS(H_0)$ is infinite.

Now consider the graph H as in (20); let for definiteness $VH = \{s_1, \ldots, s_6\}$ and $EH = \{s_is_j : 1 \le i < j \le 3 \text{ or } 4 \le i < j \le 6\}$. In order to prove that $MS(H)$ is finite take a connected graph G with $VG \supseteq VH$ and a nonnegative function l on EG. Add to G new vertices $t_1, \ldots, t_6$ and the edges $s_i t_i$, $i = 1, \ldots, 6$, forming the graph G'. Let H' be the graph with the vertex set $\{t_1, \ldots, t_6\}$ and the edge set $\{t_i t_j : 1 \le i < j \le 3$ or $4 \le i < j \le 6\}$; then H' is isomorphic to H. Take $\tau > 0$ such that $\tau d_l(pq) \le 1/2$ for all $pq \in EH$. For $i = 1, \ldots, 6$, let $J_i := \{j : s_i s_j \in EH\}$. Define the function l' on EG' by

$$\begin{aligned} l'(e) &:= \tau l(e) \qquad\qquad \text{for } e \in EG, \\ &:= \frac{1}{2}(1 + \tau d_l(s_j s_k) - \tau d_l(s_i s_j) - \tau d_l(s_i s_k)) \\ &\qquad\qquad\qquad\qquad\qquad \text{for } e = s_i t_i,\ i = 1, \ldots, 6,\ \{j, k\} = J_i. \end{aligned}$$

Then $l' \ge 0$. It is easy to check that

$$d_{l'}(pq) = \tau d_l(pq) \quad \text{for all } pq \in EH; \tag{21}$$

$$d_{l'}(p'q') = 1 \quad \text{for all } p'q' \in EH'. \tag{22}$$

Introduce the metric d^Γ of distances in the following special graph Γ (this metric occurred in [8,9]). Γ consists of 16 vertices $p_1, \ldots, p_6, x_{ij}$ ($1 \le i \le 3$, $4 \le j \le 6$) and v, and of 27 edges $p_i x_{ij}$, $p_j x_{ij}$ and $x_{ij} v$, $1 \le i \le 3$, $4 \le j \le 6$. In [9] the following statement (which is a particular case of a general theorem) was proved: if H' is the graph as above and the function l' satisfies (22), then there is a mapping $\sigma : VG' \to V\Gamma$ such that $\sigma(t_i) = p_i$, $i = 1, \ldots, 6$, and the metric m^σ H'-decomposes l'. Here $m^\sigma(xy) := d^\Gamma(\sigma(x)\sigma(y))$, $x, y \in VG'$, and we say that m' *H'-decomposes* l' if $l'(e) - \lambda m'(e) \ge 0$ for all $e \in EG'$ and $d_{l' - \lambda m'}(pq) = d_{l'}(pq) - \lambda m'(pq)$ for all $pq \in EH'$ and some $\lambda > 0$. This easily implies that for some finite k there exists an H'-packing $\{\lambda_1 m^{\sigma_1}, \ldots, \lambda_k m^{\sigma_k}\}$, where σ_i is some mapping as above. Then, by (21), $\{m_1, \ldots, m_k\}$ is an H-packing for l, where m_i is the restriction of the metric $\tau \lambda_i m^{\sigma_i}$ on VG. Thus, the cardinality of $MS(H)$ does not exceed the number of primitive metrics, each being a restriction of the metric d^Γ on a subset in $V\Gamma$. Therefore $MS(H)$ is finite.

A conjecture is: if H is as in (20) and l is cyclically even, then there exists an H-packing $\{\lambda_1 m_1, \ldots, \lambda_k m_k\}$ for l such that $m_i = m^{\sigma_i}$ for some $\sigma_i : VG \to V\Gamma$ and all λ_i are multiple of $\frac{1}{2}$.

4. A relation of H-packings to multicommodity flows

Consider a connected graph G, a graph H with $VH \supseteq VG$, and functions $l : EG \to \mathbb{R}_+$ (a *capacity* function) and $g : EH \to \mathbb{R}_+$ (a *demand* function). For $st \in EH$, denote by $\mathcal{P}_{st}$ the set of simple paths in G connecting s and t, and let $\mathcal{P} := \cup(\mathcal{P}_{st} : st \in EH)$. The multicommodity flow problem $F(G, H, c, g)$ (in the so-called "edge-path" form) is: find a function (*multicommodity flow*) $f : \mathcal{P} \to \mathbb{R}_+$ so that:

$$\sum (f(P) : P \in \mathcal{P},\ e \in P) \le c(e) \quad \text{for } e \in EG; \tag{23}$$

and

$$\sum(f(P) : P \in \mathcal{P}_{st}) = g(st) \quad \text{for } st \in EH; \tag{24}$$

or establish that such a function does not exist.

By Farkas lemma, (23)–(24) is solvable if and only if, for any $l : EG \to \mathbb{R}_+$ and $b : EH \to \mathbb{R}$, the inequality $c \cdot l \geq g \cdot b$ holds provided that

$$\sum(l(e) : e \in P) \geq b(st) \quad \text{for } P \in \mathcal{P}_{st},\ st \in EH, \tag{25}$$

where $a \cdot a'$ denotes the inner product of vectors a and a'. (25) is equivalent to $b(st) \leq d_l(st)$ for $st \in EH$. So we have the following:

(4.1) *$F(G, H, c, g)$ is solvable (i.e., a required f exists) if and only if*

$$c \cdot l \geq \sum(g(st)d_l(st) : st \in EH) \tag{26}$$

holds for any $l : EG \to \mathbb{R}_+$.

Now suppose we know that there is a set $S = \{m_1, \ldots, m_N\}$ of metrics on VG such that for any $l : EG \to \mathbb{R}_+$ there exists a *fractional* H-packing for l using metrics of S, i.e.

$$\lambda_1 m_1(e) + \ldots + \lambda_N m_N(e) \leq l(e) \quad \text{for } e \in EG, \tag{27}$$

and

$$\lambda_1 m_1(st) + \ldots + \lambda_N m_N(st) = d_l(st) \quad \text{for } st \in EH \tag{28}$$

hold for some $\lambda_1, \ldots, \lambda_N \geq 0$. Considering l as above we have from (27) and (28) that

$$c \cdot l \geq \sum_{i=1}^{N}(\lambda_i \sum_{e \in EG} c(e)m_i(e)) \tag{29}$$

and

$$\sum_{st \in EH} g(st)d_l(st) = \sum_{i=1}^{N}(\lambda_i \sum_{e \in EG} g(st)m_i(st)). \tag{30}$$

Comparing (29) and (30) with (26) we obtain the following:

(4.2) *$F(G, H, c, g)$ is solvable if and only if for any $l : EG \to \mathbb{R}_+$ the inequality*

$$\sum(c(e)m(e) : e \in EG) \geq \sum(g(st)m(st) : st \in EH) \tag{31}$$

holds for each $m \in S$. •

(The "only if" part of (4.2) follows from (4.1) if we take as l the restriction of $m \in S$ on EG.) Since arguments above can be reversed, we obtain the following relation between H-packings and multicommodity flows, mentioned in the Introduction:

(4.3) *Let G and H be graphs as above, and let S be a set of metrics on VG. The following are equivalent:*

(i) *for any c and g, problem $F(G,H,c,g)$ is solvable if and only if (31) holds for each $m \in S$;*

(ii) *for any $l : EG \to \mathbb{R}_+$, there exists a fractional H-packing for l using metrics in S.* •

For example, (4.3) enables us to derive the statement (1.2) directly from the theorem of Papernov [12] (and vice versa): if H is K_4 or C_5 or a union of two stars, then $F(G,H,c,g)$ is solvable if and only if (31) holds for all cut metrics m on VG. Similarly, a weaker, "fractional", version of Theorem 1 derives the following fractional version of a theorem in [8]: if $H = K_5$ then $F(G,H,c,g)$ is solvable if and only if (31) holds for each cut metric and each 2,3-metric on VG. Note that linear programming duality arguments as above gives relations only between corresponding "fractional" problems and they, of course, are not sufficient to derive half-integral H-packing theorems, such as Theorem 1 or (1.3), as well as half-integral multicommodity flow theorems, such as in [8] or in [11].

REFERENCES

[1] P. Assouad and M. Deza, Metric subspaces of L^1, *Publications Mathematiques d'Orsay*, v. **3**, 1982, 47pp.

[2] D. Avis, On the extreme rays of the metric cone, *Canadian J. of Math.* **32** (1980) 126–144.

[3] V. Chvatal, Recognizing intersection patterns, *Ann. of Discrete Math.*, **8** (1980) 249–251.

[4] M. Deza, On Hamming geometry of unit cubes, *Doklady Akademii Nauk SSSR* **134** (1960) 1037–1040 (in Russian).

[5] D.Z. Djokovic, Distances preserving subgraphs of hypercubes, *J. Comb. Theory B* **14** (1973) 263–267.

[6] M. Grotschel and L. Lovasz, and A.Schrijver, *Geometric Algorithms and Combinatorial Optimization* (Springer, 1987) 340pp.

[7] A.V. Karzanov, Metrics and undirected cuts, *Math. Programming* **32** (1985) 183-198.

[8] A.V. Karzanov, Half-integral five-terminus flows, *Discrete Appl. Math.* **18** (3) (1987) 263–278.

[9] A.V. Karzanov, Polyhedra related to undirected multicommodity flows, *Linear Algebra and Appl.* **114/115** (1989) 293–328.

[10] M.V. Lomonosov, On a system of flows in a network, *Problemy Peredachi Infor-*

matsii **14** (1978) 60–73 (in Russian).

[11] M.V. Lomonosov, Combinatorial approaches to multiflow problems, *Discrete Appl. Math.* **11** (1) (1985) 1–94.

[12] B.A. Papernov, On existence of multicommodity flows, in: *Studies in Discrete Optimizations* (A.A. Fridman, ed., Nauka, Moscow, 1976) pp. 230–261, in Russian.

[13] A. Schrijver, Short proofs on multicommodity flows and cuts, to appear.