Path systems in certain planar graphs and quadratic identities for quantum minors

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Abstract

We deal with weighted planar graphs G, called *grid-shaped* ones, arising as a natural generalization of the graphs associated to Cauchon diagrams. Adapting an approach due to Casteels [2], we consider a matrix Path_G generated by paths in G, whose entries are shown to obey quasicommutation relations similar to those in quantum matrices. By a version of Lindström Lemma, certain systems or disjoint paths, or *flows*, in G generate minors of Path_G.

This paper gives a complete combinatorial characterization of homogeneous quadratic relations of "universal character" valid for minors of the matrix Path_G of any grid-shaped graph G and, as a consequence, for minors in quantized coordinate rings of matrices over a field. For this purpose, we develop a quantum version of the *flow-matching method* elaborated for the commutative case in [5]. The method is illustrated with representative examples and an algorithm of recognizing universal quadratic identities for quantum minors is devised.

Keywords: quantum matrix, quantum affine space, quadratic identity for minors, planar graph, Cauchon diagram, Lindström Lemma

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1 Introduction

The idea of quantization of certain algebraic structures has proved its importance to bridge the commutative and noncommutative versions of the structure and promote better understanding various aspects of the latter. On this way one of the most popular structures studied in the last three decades is the quantized coordinate ring $\mathcal{R} = \mathcal{O}_q(\mathcal{M}_{m,n}(\mathbb{K}))$ of $m \times n$ matrices over a field $\mathbb{K}$, where q is a nonzero element of $\mathbb{K}$, usually called the *algebra of $m \times n$ quantum matrices*. Here we deal with an $m \times n$ matrix X of indeterminates x_{ij}, and $\mathcal{R}$ is the $\mathbb{K}$-algebra generated by the x_{ij} subject to the following (quasi)commutation relations: for $i < \ell \le m$ and $j < k \le n$,

$$\begin{gathered} x_{ij}x_{ik} = qx_{ik}x_{ij}, \qquad x_{ij}x_{\ell j} = qx_{\ell j}x_{ij}, \\ x_{ik}x_{\ell j} = x_{\ell j}x_{ik} \quad \text{and} \quad x_{ij}x_{\ell k} - x_{\ell k}x_{ij} = (q - q^{-1})x_{ik}x_{\ell j}, \end{gathered} \tag{1.1}$$

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called Manin's relations [14].

This paper is devoted to quadratic identities for minors of quantum matrices, or *quantum minors*. For an important discussion on representative cases, various aspects and applications of such identities, see [8, 9, 10, 11, 15] (where the list is incomplete). We present a novel, and rather transparent, combinatorial method which enables us to completely characterize and efficiently verify homogeneous quadratic identities of universal character that are valid for quantum minors.

The identities of our interest can be written as

$$\sum(\mathrm{sign}_i q^{\delta_i}[I_i|J_i]_q\,[I'_i|J'_i]_q\colon i=1,\dots,N)=0, \tag{1.2}$$

where $\delta_i\in\mathbb{Z}$, $\mathrm{sign}_i\in\{+,-\}$, and $[I|J]_q$ denotes the quantum minor whose rows and columns are indexed by $I\subseteq\{1,\dots,m\}$ and $J\subseteq\{1,\dots,n\}$, respectively. The homogeneity means that the sets $I_i\cup I'_i$, $I_i\cap I'_i$, $J_i\cup J'_i$, $J_i\cap J'_i$ are invariant of i, and the term "universal" means that (1.2) should be valid independently of $\mathbb{K},q$ and the q-matrix M in question (i.e., assuming that M is an $m\times n$ matrix of any formation whose entries obey relations similar to those in (1.1) and, possibly, additional relations). Note that any cortege $(I|J,I'|J')$ may be repeated in (1.2) many times.

Our approach is based on two sources. First, it is a far modification, to work with the more sophisticated quantum situation, of the *flow-matching* method elaborated in [5] to characterize quadratic identities for usual minors, i.e., in the commutative case. (In that case the identities are viewed simpler than (1.2), namely, as

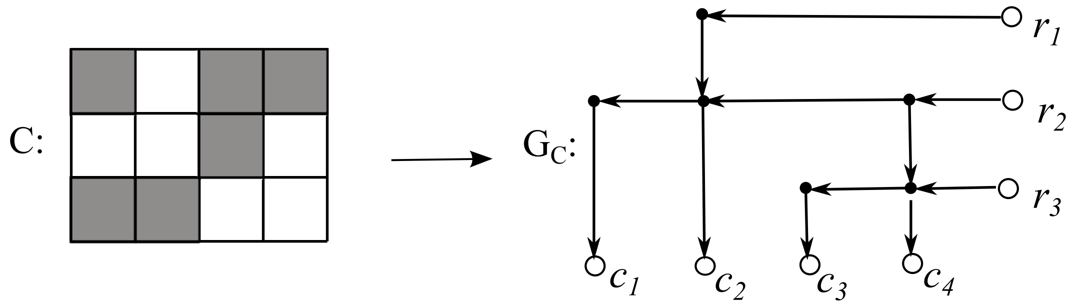
$$\sum(\mathrm{sign}_i[I_i|J_i]\,[I'_i|J'_i]\colon i=1,\dots,N)=0, \tag{1.3}$$

Also there are natural analogues of (1.3) over commutative semirings, e.g. the tropical semiring $(\mathbb{R},+,\max)$.)

In the method of [5], each cortege $S=(I|J,I'|J')$ is associated with a certain set $\mathcal{M}(S)$ of feasible matchings on $(I\triangle I')\sqcup(J\triangle J')$ (where $A\triangle B$ denotes the symmetric difference, and $A\sqcup B$ the disjoint union of sets A,B). The main theorem in [5] asserts that (1.3) is valid (universally) if and only if the families $\mathcal{I}^+$ and $\mathcal{I}^-$ of corteges S_i with $\mathrm{sign}_i=+$ and $\mathrm{sign}_i=-$, respectively, are *balanced*, in the sense that the total families of feasible matchings for corteges occurring in $\mathcal{I}^+$ and in $\mathcal{I}^-$ are equal.

The main result of this paper gives necessary and sufficient conditions for the quantum version (in Theorems 7.1 and 5.1), by saying that (1.2) is valid (universally) if and only if the families of corteges $\mathcal{I}^+$ and $\mathcal{I}^-$ along with the function δ are *q-balanced*, which now means the existence of a bijection between the feasible matchings for $\mathcal{I}^+$ and $\mathcal{I}^-$ that is agreeable with δ in a certain sense. The proof of necessity (Theorem 7.1) considers non-q-balanced $\mathcal{I}^+,\mathcal{I}^-,\delta$ and explicitly constructs a certain graph generating a q-matrix for which (1.2) is violated when $\mathbb{K}$ is a field of characteristic 0 and q is transcendental over $\mathbb{Q}$.

The second source of our approach is the path method due to Casteels [2, 3]. He associated to each Cauchon diagram C of [1] a directed planar graph $G=G_C$ with m row-representing vertices (sources) $r_1,\dots,r_m$ and n column-representing vertices (sinks) $c_1, \dots, c_n$, whose inner vertices correspond to white cells (i,j) in the diagram C and are labeled as t_{ij}. An example is illustrated in the picture.



The labels t_{ij}, regarded as indeterminates, (quasi)commute as

$$\begin{aligned} t_{ij}t_{i'j'} &= qt_{i'j'}t_{ij} \qquad \text{if either } i=i' \text{ and } j<j', \text{ or } i<i' \text{ and } j=j', \\ &= t_{i'j'}t_{ij} \qquad \text{otherwise} \end{aligned} \tag{1.4}$$

(which is "simpler" than (1.1)). They are used to define weights of edges and, further, weights of paths of G, giving Laurent monomials in the t_{ij}. This leads to forming the so-called *path matrix* of size $m \times n$, denote it as Path_G, of which (i,j)-th entry is the sum of weights of paths starting at r_i and ending at c_i.

The path matrix $\mathrm{Path}_G = (m_{ij})$ has three important properties. (i) It is a q-matrix, and therefore, $x_{ij} \mapsto m_{ij}$ gives a homomorphism of $\mathcal{R}$ to the corresponding algebra $\mathcal{R}_G$ generated by the m_{ij}. (ii) Path_G admits an analogue of Lindström's Lemma [13]: for any $I \subseteq \{1,\dots,m\}$ and $J \subseteq \{1,\dots,n\}$ with $|I|=|J|$, the minor $[I|J]_q$ of Path_G can be expressed as the sum of weights of systems of *disjoint paths* from $\{r_1 \colon i \in I\}$ to $\{c_j \colon j \in J\}$ in G. (iii) From Cauchon's Algorithm [1] interpreted in graph terms in [2, 3] it follows that: if the diagram C is maximal (i.e., has no black cells), then Path_G becomes a *generic q-matrix*, see Corollary 3.2.5 in [3].

We consider a more general class of planar graphs G (with horizontal and vertical edges), called *grid-shaped* ones, and show that they satisfy the above properties (i)–(iii). Our goal is to characterize quadratic identities just for the class of path matrices Path_G of grid-shaped graphs G. Since this class contains a generic q-matrix, the identities are automatically valid in $\mathcal{R}$.

We take an advantage from the fact that the minors are expressed via systems of disjoint paths, or *flows* in our terminology, and the desired results are obtained by applying a combinatorial machinery of handling flows in grid-shaped graphs (which is a development of that in [5]). Our method of establishing or verifying one or another identity is illustrated by enlightening graphical diagrams.

The paper is organized as follows. Section 2 contains basic definitions and backgrounds. Section 3 defines flows and path matrices for grid-shaped graphs and states Lindström's type theorem for them. Section 4 is devoted to crucial ingredients of the method. It describes *exchange operations* on *double flows* (pairs of flows related to corteges $(I|J, I'|J')$) and expresses such operations on the language of planar matchings. The main working tool of the whole proof, stated in this section and proved in a separate paper, is Theorem 4.4 giving a q-relation between double flows before and after an ordinary exchange operation. Using this, Section 5 proves the sufficiency in the main result: (1.2) is valid if the corresponding $\mathcal{I}^+, \mathcal{I}^-, \delta$ are q-balanced (Theorem 5.1).

Section 6 is devoted to illustrations of our method. It explains how to obtain, with the help of the method, rather transparent proofs for several representative examples of quadratic identities, in particular: (a) the pure commutation of $[I|J]_q$ and $[I'|J']_q$ when $I' \subset I$ and $J' \subset J$; (b) a quasicommutation of flag minors $[I]_q$ and $[J]_q$ as in Leclerc-Zelevinsky's theorem [12]; (c) identities on flag minors involving triples $i < j < k$ and quadruples $i < j < k < \ell$; (d) Dodgson's type identity; (e) two general quadratic identities on flag minors from [11, 15] occurring in descriptions of quantized Grassmannians and flag varieties. In Section 7 we prove the necessity of the q-balancedness condition for validity of quadratic identities (Theorem 7.1); here we rebuild a corresponding construction from [5] to obtain, in case of the non-q-balancedness, a grid-shaped graph G such that the identity for Path_G is false (in a special case of $\mathbb{K}$ and q). The concluding Section 8 poses the problem: when an identity in the commutative case, such as (1.3), can be turned, by choosing an appropriate δ, into the corresponding identity for the quantized case? For example, this is impossible for the trivial identity $[I]\,[J] = [J]\,[I]$ with usual flag minors when I, J are not weakly separated, as is shown in [12]. We finish the paper with a generalization of Leclerc-Zelevinsky's quasicommutation theorem to arbitrary (non-flag) quantum minors (Theorem 8.1).

2 Preliminaries

2.1 Paths in graphs. Throughout, by a *graph* we mean a directed graph. A *path* in a graph $G = (V, E)$ (with vertex set V and edge set E) is a sequence $P = (v_0, e_1, v_1, \dots, e_k, v_k)$ such that each e_i is an edge connecting vertices v_{i-1}, v_i. An edge e_i is called *forward* if it is directed from v_{i-1} to v_i, denoted as $e_i = (v_{i-1}, v_i)$, and *backward* otherwise (when $e_i = (v_i, v_{i-1})$). The path P is called *directed* if it has no backward edge, and *simple* if all vertices v_i are different. When $k > 0$ and $v_0 = v_k$, P is called a *cycle*, and called a *simple cycle* if, in addition, $v_1, \dots, v_k$ are different. When it is not confusing, we may use for P the abbreviated notation via vertices: $P = v_0 v_1 \dots v_k$, or edges $P = e_1 e_2 \dots e_k$.

We will often deal with the situation when the edges $e \in E$ are equipped with *weights* $w(e)$ taken from a non-commutative ring. Then the weight $w(P)$ of a path $P = e_1 e_2 \dots e_k$ is the ordered (from left to right) product

$$w(P) = w(e_1) w(e_2) \cdots w(e_k). \tag{2.1}$$

2.2 Quantum matrices. For a positive integer n, the set $\{1, 2, \dots, n\}$ is denoted by $[n]$.

It will be convenient for us to visualize matrices in the Cartesian form: for an $m \times n$ matrix $A = (a_{ij})$, the row indices $i = 1, \dots, m$ are assumed to increase upwards, and the column indices $j = 1, \dots, n$ from left to right.

Fix a field $\mathbb{K}$ and an element $q \in \mathbb{K}^*$, and consider the $m \times n$ matrix with indeterminates x_{ij}. The *quantized coordinate ring* $\mathcal{R} = \mathcal{O}_q(\mathcal{M}_{m,n}(\mathbb{K}))$ is the $\mathbb{K}$-algebra generated by the x_{ij} satisfying relations (1.1). Such an $\mathcal{R}$ is shortly called the algebra of $m \times n$ *quantum matrices.*

A somewhat "simpler" object to study is the $m \times n$ *quantum affine space* $\overline{\mathcal{R}}$, the $\mathbb{K}$-algebra generated by indeterminates t_{ij} ($i \in [m]$, $j \in [n]$) subject to relations (1.4).

2.3 Quantum minors. For an $m \times n$ matrix $M = (m_{ij})$, we denote by $M(I|J)$ the submatrix of M whose rows are indexed by $I \subseteq [m]$, and columns indexed by $J \subseteq [n]$. Let $|I| = |J| =: k$, I consist of $i_1 < \dots < i_k$, and J consist of $j_1 < \dots < j_k$. Then the *q-determinant* of $M(I|J)$, or the *q-minor* of M for $I|J$, is defined as

$$[I|J]_{M,q} := \sum_{\sigma \in S_k} (-q)^{\ell(\sigma)} \prod_{d=1}^{k} m_{i_d j_{\sigma(d)}}, \tag{2.2}$$

where, in the noncommutative case, the product under $\prod$ is ordered by increasing d, and $\ell(\sigma)$ denotes the *length* (number of inversions) of a permutation σ. The terms M and/or q in $[I|J]_{M,q}$ may be omitted when they are clear from the context.

In particular, when dealing with quantum matrices as in Section 2.2, $[I|J]$ is the corresponding polynomial in the x_{ij} of degree k.

2.4 Grid-shaped graphs. Such a graph $G = (V, E)$ (also denoted as $(V, E; R, C)$) satisfies the following conditions:

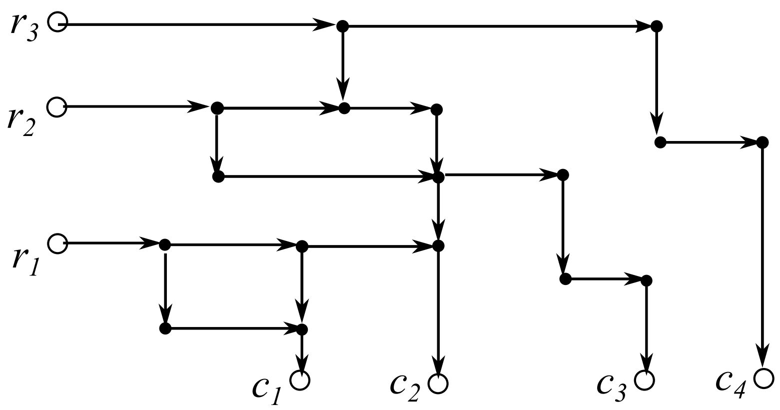
(i) G is planar (with a fixed layout in the plane);

(ii) G has edges of two types: *horizontal* edges, or *H-edges*, which are directed to the right, and *vertical* edges, or *V-edges*, which are directed downwards;

(iii) G has two distinguished subsets of vertices: set $R = \{r_1, \dots, r_m\}$ of *sources* and set $C = \{c_1, \dots, c_n\}$ of *sinks*; moreover, $r_1, \dots, r_m$ are disposed on a vertical line, in this order upwards, and $c_1, \dots, c_n$ are disposed on a horizontal line, in this order from left to right;

(iv) each vertex (and each edge) of G belongs to a directed path from R to C.

The set $V - (R \cup C)$ if *inner* vertices of a grid-shaped graph $G = (V, E)$ is denoted by $W = W_G$. An example of grid-shaped graphs with $m = 3$ and $n = 4$ is drawn in the picture:



Remark 1. In fact, grid-shaped graphs generalize *Cauchon graphs*, mentioned in the Introduction, up to two differences: in the definition of a Cauchon graph in [2], the horizontal edges are directed from right to left, and the "row-representing vertices" $r_1, \dots, r_m$ (which are the rightmost vertices of the graph) are ordered downwards.

An especial role is played by the Cauchon graph (for m, n) generated by the diagram without black cells, i.e., when the set W of inner vertices is $[n] \times [m]$, and the inner vertices $u = (i, j)$ and $v = (i', j')$ are connected by edge from u to v if either $i' = i + 1$ and $j = j'$, or $i = i'$ and $j' = j - 1$ (using the Cartesian coordinates and increasing the row indices upward). We refer to such a graph as the *extended* (m, n)*-grid* and denote it as $\Gamma(m, n)$.

Each inner vertex $v \in W$ is regarded as an indeterminate (generator), and we assign weights $w(e)$ to the edges $e \in E$ in a way similar to that for Cauchon graphs in [2]. More precisely, for an edge $e = (u, v)$,

(2.3) (i) $w(e) := v$ if $u \in R$;

(ii) $w(e) := u^{-1}v$ if e is an H-edge and $u, v \in W$;

(iii) $w(e) := 1$ if e is a V-edge.

Then the weight $w(P)$ of any path P in G is defined as in (2.1).

Remark 2. When P is a directed path from R to C, the weight $w(P)$ can also be expressed in the following useful form; cf. [3, Prop. 3.1.8]. Let $u_1, v_1, u_2, v_2, \dots, u_{d-1}, v_{d-1}, u_d$ be the sequence of vertices where P turns; namely, P changes the horizontal direction to the vertical one at each u_i, and conversely at each v_i. Then, due to the "telescopic effect",

$$w(P) = u_1 v_1^{-1} u_2 v_2^{-1} \cdots u_{d-1} v_{d-1}^{-1} u_d.$$

We assume that the indeterminates W obey (quasi)commutation laws somewhat similar to those in (1.4); namely,

(2.4) for distinct $u, v \in W$,

(i) if there is a directed *horizontal* path from u to v in G, then $uv = qvu$;

(ii) if there is a directed *vertical* path from u to v in G, then $vu = quv$;

(iii) otherwise $uv = vu$.

(This matches the commutation behavior for Cauchon graphs in [2] where the inner vertices represent the corresponding "white-colored" generators t_{ij} in the quantum affine space $\overline{\mathcal{R}}$.)

3 Path matrix and flows

An important construction in [2] associates to a Cauchon graph G a certain matrix, called the path matrix of G, which has a nice property of Lindström's type, saying that minors of this matrix correspond to appropriate systems of disjoint paths in G.

This is extended to an arbitrary grid-shaped graph $G = (V, E)$. More precisely, let m and n be the numbers $|R|$ and $|C|$ of sources and sinks of G, respectively. As before, $w = w_G$ denotes the edge weights in G defined by (2.3).

Definition. The *path matrix* $\mathrm{Path} = \mathrm{Path}_G$ associated to G is the $m \times n$ matrix whose entries are defined by

$$\mathrm{Path}(i|j) := \sum\nolimits_{P \in \Phi_G(i|j)} w(P), \qquad (i,j) \in [m] \times [n], \tag{3.1}$$

where $\Phi_G(i|j)$ is the set of directed paths from r_i to c_j in G. In particular, $\mathrm{Path}(i|j) = 0$ if $\Phi_G(i|j) = \emptyset$.

Thus, the entries of Path_G belong to the $\mathbb{K}$-algebra $\mathcal{L}_G$ of Laurent polynomials generated by the set W of inner vertices of G subject to relations (2.4).

Definition. Let $\mathcal{E}^{m,n}$ denote the set of pairs $(I|J)$ such that $I \subseteq [m]$, $J \subseteq [n]$ and $|I| = |J|$. Borrowing terminology from [5], for $(I|J) \in \mathcal{E}^{m,n}$, a set ϕ of pairwise disjoint directed paths from the source set $R_I := \{r_i \colon i \in I\}$ to the sink set $C_J := \{c_j \colon j \in J\}$ in G is called an *$(I|J)$-flow*.

The set of $(I|J)$-flows ϕ in G is denoted by $\Phi(I|J) = \Phi_G(I|J)$. We usually assume that the paths forming ϕ are ordered by increasing the source indices. Namely, if I consists of $i(1) < i(2) < \cdots < i(k)$ and J consists of $j(1) < j(2) < \cdots < j(k)$, then ℓ-th path P_ℓ in ϕ begins at $r_{i(\ell)}$, and therefore, P_ℓ ends at $c_{j(\ell)}$ (which easily follows from the planarity of G, the ordering of sources and sinks in the boundary of G and the fact that the paths in ϕ are disjoint). We write $\phi = (P_1, P_2, \dots, P_k)$ and (similar to path systems in [2]) define the weight of ϕ to be the ordered product

$$w(\phi) = w(P_1)w(P_2)\cdots w(P_k). \tag{3.2}$$

This gives rise to the following function $f = f_G$ on $\mathcal{E}^{m,n}$ taking values in the algebra $\mathcal{L}_G$:

$$f(I|J) := \sum\nolimits_{\phi \in \Phi(I|J)} w(\phi), \qquad (I|J) \in \mathcal{E}^{m,n}. \tag{3.3}$$

We call f_G a *flow-generated* function. Slightly generalizing a q-analogue of Lindström's Lemma from [2, Th. 4.4], one can express f_G via minors of the path matrix Path_G.

Theorem 3.1 *For each $(I|J) \in \mathcal{E}^{m,n}$, $f(I|J)$ is equal to $[I|J]_{\mathrm{Path},q}$.*

A proof of this theorem, which is close to that in [2], is given in [4].

An important fact is that the (quasi)commutation relations for the entries of Path_G are similar to those for the canonical generators x_{ij} of the quantum algebra $\mathcal{R}$ given in (1.1). It is exhibited in the following assertion, which is known for the path matrices of Cauchon graphs due to [2] (where it is proved by use of the "Cauchon's deleting derivation algorithm in reverse" [1]).

Theorem 3.2 *For a grid-shaped graph G, any 1×2 submatrix $(a\ b)$ of the matrix Path_G satisfies $ab = qba$. Any 2×1 submatrix $\binom{c}{a}$ of Path_G satisfies $ac = qca$. Any 2×2 submatrix $\binom{c\ d}{a\ b}$ of Path_G satisfies $bc = cb$ and $ad - da = (q - q^{-1})bc$.*

We will show this in Section 6.3 as an easy application of our flow-matching method. This assertion implies that the map $x_{ij} \mapsto \mathrm{Path}_G(i|j)$ determines a homomorphism of $\mathcal{R}$ to the subalgebra $\mathcal{R}_G$ of $\mathcal{L}_G$ generated by the entries of Path_G, i.e., Path_G is a q-matrix for any grid-shaped graph G. In an especial case of G, a sharper result, attributed to Cauchon and Casteels, is as follows.

Theorem 3.3 ([1, 3]) *If $G = \Gamma(m, n)$ (the extended $m \times n$-grid defined in Section 2.4), then Path_G is a generic q-matrix, i.e., $x_{ij} \mapsto \mathrm{Path}_G(i|j)$ gives an injective map of $\mathcal{R}$ to $\mathcal{L}_G$.*

Due to this important property, the quadratic relations that will be proved to be valid (universally) for minors of path matrices of grid-shaped graphs turn out to be automatically valid for the algebra $\mathcal{R}$ of quantum matrices, and vice versa.

4 Double flows, matchings, and exchange operations

Quadratic identities of our interest in this paper involve products of the form $f(I|J)f(I'|J')$, where $(I|J), (I'|J') \in \mathcal{E}^{m,n}$, and f is defined by (3.3). This causes to a proper study of ordered pairs of flows $\phi \in \Phi(I|J)$ and $\phi' \in \Phi(I'|J')$.

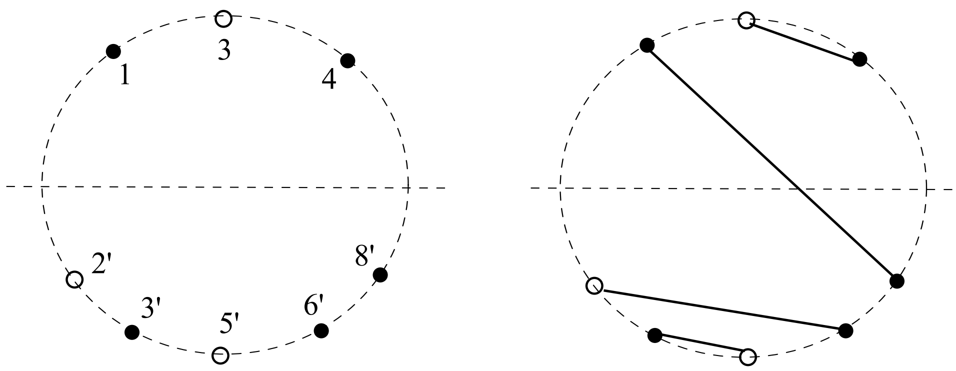
We need some definitions and conventions, borrowing terminology from [5]. Given $I, J, I', J', \phi, \phi'$ as above, we call the pair (ϕ, ϕ') a *double flow* in G. Let

$$I^\circ := I - I', \quad J^\circ := J - J', \quad I^\bullet := I' - I, \quad J^\bullet := J' - J, \qquad (4.1)$$
$$Y^{\mathrm{r}} := I^\circ \cup I^\bullet \quad \text{and} \quad Y^{\mathrm{c}} := J^\circ \cup J^\bullet.$$

Note that $|I| = |J|$ and $|I'| = |J'|$ imply that $|Y^{\mathrm{r}}| + |Y^{\mathrm{c}}|$ is even and that

$$|I^\circ| - |I^\bullet| = |J^\circ| - |J^\bullet|. \qquad (4.2)$$

It will be convenient for us to interpret I° and $I^\bullet$ as the sets of *white* and *black* elements of $Y^{\rm r}$, respectively, and similarly for $J^\circ, J^\bullet, Y^{\rm c}$, and to visualize these objects by use of a *circular diagram* D in which the elements of $Y^{\rm r}$ (resp. $Y^{\rm c}$) are disposed in the increasing order from left to right in the upper (resp. lower) half of a circumference O. For example if, say, $I^\circ=\{3\}$, $I^\bullet=\{1,4\}$, $J^\circ=\{2',5'\}$ and $J^\bullet=\{3',6',8'\}$, then the diagram is viewed as in the left fragment of the picture below. (Sometimes, to avoid a possible mess between elements of $Y^{\rm r}$ and $Y^{\rm c}$, and when it leads to no confusion, we denote elements of $Y^{\rm c}$ with primes.)



Definition. We refer to the quadruple $(I|J,I'|J')$ as above as a *cortege*, and to $(I^\circ,I^\bullet,J^\circ,J^\bullet)$ as the *refinement* of $(I|J,I'|J')$, or as a *refined cortege*.

Let M be a partition of $Y^{\rm r}\sqcup Y^{\rm c}$ into 2-element sets (recall that $A\sqcup B$ denotes the disjoint union of sets A,B). We refer to M as a *perfect matching* on $Y^{\rm r}\sqcup Y^{\rm c}$, and to its elements as *couples*.

Also we say that $\pi\in M$ is: an *R-couple* if $\pi\subseteq Y^{\rm r}$, a *C-couple* if $\pi\subseteq Y^{\rm c}$, and an *RC-couple* if $|\pi\cap Y^{\rm r}|=|\pi\cap Y^{\rm c}|=1$ (as though π "connects" two sources, two sinks, and one source and one sink, respectively).

Definition. A (perfect) matching M as above is called a *feasible* matching for $(I^\circ,I^\bullet,J^\circ,J^\bullet)$ (and for $(I|J,I'|J')$) if:

(4.3) (i) for each $\pi=\{i,j\}\in M$, the elements i,j have different colors if π is an R- or C-couple, and have the same color if π is an RC-couple;

(ii) M is *planar*, in the sense that the chords connecting the couples in the circumference O are pairwise non-intersecting.

The set of feasible matchings for $(I^\circ,I^\bullet,J^\circ,J^\bullet)$ is denoted by $\mathcal{M}_{I^\circ,I^\bullet,J^\circ,J^\bullet}$ and may also be denoted as $\mathcal{M}(I|J,I'|J')$. This set is nonempty whenever $Y^{\rm r}\sqcup Y^{\rm c}\ne\emptyset$. (Indeed, a feasible matching can be constructed recursively as follows. Let for definiteness $|I^\circ|\ge|I^\bullet|$. If $I^\bullet\ne\emptyset$, then choose $i\in I^\circ$ and $j\in I^\bullet$ with $|i-j|$ minimum, form the R-couple $\{i,j\}$ and delete i,j. And so on until $I^\bullet$ becomes empty. Act similarly for J° and $J^\bullet$. Eventually, in view of (4.2), we obtain $I^\bullet=J^\bullet=\emptyset$ and $|I^\circ|=|J^\circ|$. Then we form corresponding white RC-couples.)

The right fragment of the above picture illustrates an instance of feasible matchings.

Return to a double flow (ϕ, ϕ') as above. Our aim is to associate to it a feasible matching for $(I^\circ, I^\bullet, J^\circ, J^\bullet)$.

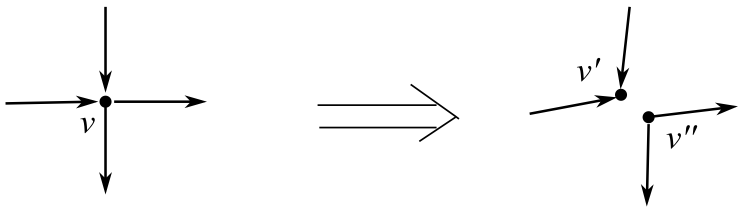
To do this, we write V_ϕ and E_ϕ, respectively, for the sets of vertices and edges of G occurring in ϕ, and similarly for ϕ'. An important role will be played by the subgraph $\langle U \rangle$ of G induced by the set of edges

$$U := E_\phi \triangle E_{\phi'},$$

denoting $A \triangle B$ the symmetric difference $(A - B) \cup (B - A)$ of sets A, B. Note that

(4.4) a vertex v of $\langle U \rangle$ has degree 1 if $v \in R_{I^\circ} \cup R_{I^\bullet} \cup C_{J^\circ} \cup C_{J^\bullet}$, and degree 2 or 4 otherwise.

We slightly modify $\langle U \rangle$ by splitting each vertex v of degree 4 in $\langle U \rangle$ (if any) into two vertices v', v'' disposed in a small neighborhood of v so that the edges entering (resp. leaving) v become entering v' (resp. leaving v''); see the picture.



The resulting graph, denoted as $\langle U \rangle'$, is planar and has vertices of degree only 1 and 2. Therefore, $\langle U \rangle'$ consists of pairwise disjoint (non-directed) simple paths $P'_1, \ldots, P'_k$ (considered up to reversing) and, possibly, simple cycles $Q'_1, \ldots, Q'_d$. The corresponding images of $P'_1, \ldots, P'_k$ (resp. $Q'_1, \ldots, Q'_d$) give paths $P_1, \ldots, P_k$ (resp. cycles $Q_1, \ldots, Q_d$) in $\langle U \rangle$. When $\langle U \rangle$ has vertices of degree 4, some of the latter paths and cycles may be self-intersecting and may "touch", but not "cross", each other.

Lemma 4.1 (i) *$k = (|I^\circ| + |I^\bullet| + |J^\circ| + |J^\bullet|)/2$;*

(ii) *the set of endvertices of $P_1, \ldots, P_k$ is $R_{I^\circ \cup I^\bullet} \cup C_{J^\circ \cup J^\bullet}$; moreover, each P_i connects either R_{I° and $R_{I^\bullet}$, or C_{J° and $C_{J^\bullet}$, or R_{I° and C_{J°, or $R_{I^\bullet}$ and $C_{J^\bullet}$;*

(iii) *in each path P_i, the edges of ϕ and the edges of ϕ' have different directions (say, the former edges are all forward, and the latter ones are all backward).*

Proof (i) is trivial, and (ii) follows from (iii) and the fact that the sources r_i (resp. sinks c_j) have only outgoing (resp. incoming) edges. In its turn, (iii) easily follows by considering a common (inner) vertex v of a directed path K in ϕ and a directed path L in ϕ'. Let e, e' (resp. u, u') be the edges of K (resp. L) incident to v. Then: if $\{e, e'\} = \{u, u'\}$, then v vanishes in $\langle U \rangle$. If $e = u$ and $e' \neq u'$, then either both e', u' enter v, or both e', u' leave v; whence e', u' are consecutive and differently directed edges of some path P_i or cycle Q_j. A similar property holds when $\{e, e'\} \cap \{u, u'\} = \emptyset$, as being a consequence of splitting v into two vertices as described. ∎

Thus, each P_i is represented as a concatenation $P_i^{(1)} \circ P_i^{(2)} \circ \ldots \circ P_i^{(\ell)}$ of forwardly and backwardly directed paths which are alternately contained in ϕ and ϕ'. We call P_i an *exchange path* (by a reason that will be clear later) and call $P_i^{(j)}$ a (forward or backward) *segment* of P_i. The endvertices of P_i determine, in a natural way, a pair of elements of $Y^{\mathrm{r}} \sqcup Y^{\mathrm{c}}$, denoted by π_i. Then $M := \{\pi_1, \ldots, \pi_k\}$ is a perfect matching on $Y^{\mathrm{r}} \sqcup Y^{\mathrm{c}}$. Moreover, it is a feasible matching, since (4.3)(i) follows from Lemma 4.1(ii), and (4.3)(ii) is provided by the fact that $P'_1, \ldots, P'_k$ are pairwise disjoint simple paths in $\langle U \rangle'$.

We denote M as $M(\phi, \phi')$, and for $\pi \in M$, denote by $P(\pi)$ the exchange path P_i corresponding to π (i.e., $\pi = \pi_i$).

Corollary 4.2 $M(\phi, \phi') \in \mathcal{M}_{I^\circ, I^\bullet, J^\circ, J^\bullet}$.

Figure 1 illustrates an instance of ϕ, ϕ' for $I = \{1, 2, 3\}$, $J = \{1', 3', 4'\}$, $I' = \{2, 4\}$, $J' = \{2', 3'\}$; here ϕ and ϕ' are drawn by solid and dotted lines, respectively (in the left fragment), the subgraph $\langle E_\phi \triangle E_{\phi'} \rangle$ consists of three paths and one cycle (in the middle), and the circular diagram illustrates $M(\phi, \phi')$ (in the right fragment).

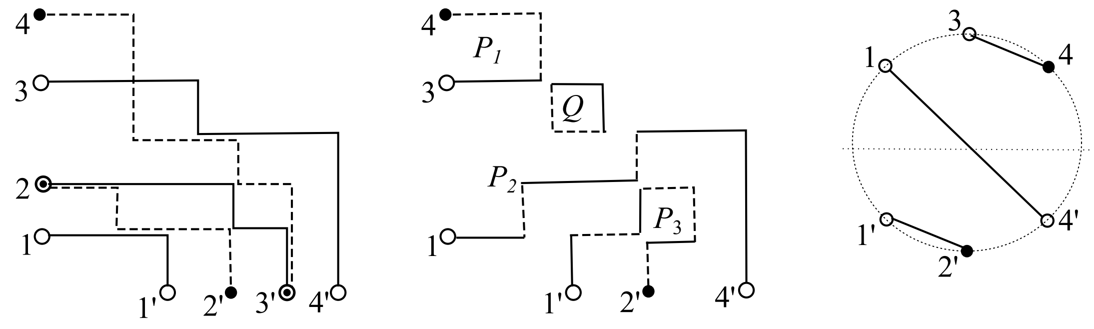


Figure 1: flows ϕ and ϕ' (left); $\langle E_\phi \triangle E_{\phi'} \rangle$ (middle); $M(\phi, \phi')$ (right)

Flow exchange operation. It rearranges a given double flow (ϕ, ϕ') for $(I|J,\, I'|J')$ into another double flow (ψ, ψ') for some $(\widetilde{I}|\widetilde{J},\, \widetilde{I}'|\widetilde{J}')$, as follows. Fix a submatching $\Pi \subseteq M(\phi, \phi')$, and combine the exchange paths concerning Π, forming the set of edges

$$\mathcal{E} := \cup(E_{P(\pi)} \colon \pi \in \Pi).$$

(where E_P denotes the set of edges in a path P).

Lemma 4.3 *Let* $V\Pi := \cup(\pi \in \Pi)$. *Define*

$$\widetilde{I} := I \triangle V\Pi, \quad \widetilde{I}' := I' \triangle V\Pi, \quad \widetilde{J} := J \triangle V\Pi, \quad \widetilde{J}' := J' \triangle V\Pi.$$

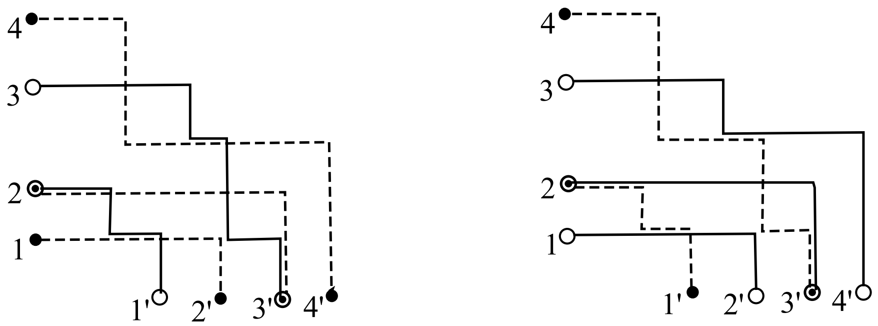
Then the subgraph ψ *induced by* $E_\phi \triangle \mathcal{E}$ *gives a* $(\widetilde{I}|\widetilde{J})$*-flow, and the subgraph* ψ' *induced by* $E_{\phi'} \triangle \mathcal{E}$ *gives a* $(\widetilde{I}'|\widetilde{J}')$*-flow in* G. *Furthermore,* $E_\psi \cup E_{\psi'} = E_\phi \cup E_{\phi'}$, $E_\psi \triangle E_{\psi'} = E_\phi \triangle E_{\phi'}$ $(= U)$, *and* $M(\psi, \psi') = M(\phi, \phi')$.

Proof Consider a path $P = P(\pi)$ for $\pi \in \Pi$, and let P consist of segments $P^{(1)}, P^{(2)}, \dots, P^{(\ell)}$. Let for definiteness the segments $P^{(d)}$ with d odd concern ϕ, and denote by v_d the common endvertex of $P^{(d)}$ and $P^{(d+1)}$. Under the operation $E_\phi \mapsto E_\phi \triangle E_P$ the pieces $P^{(1)}, P^{(3)}, \dots$ in ϕ are replaced by $P^{(2)}, P^{(4)}, \dots$. In its turn, $E_{\phi'} \mapsto E_{\phi'} \triangle E_P$ replaces the pieces $P^{(2)}, P^{(4)}, \dots$ in ϕ' by $P^{(1)}, P^{(3)}, \dots$.

By Lemma 4.1(iii), for each d, the edges of $P^{(d)}, P^{(d+1)}$ incident to v_d either both enter or both leave v_d. Also each intermediate vertex of any segment $P^{(d)}$ occurs in exactly one flow among ϕ, ϕ'. These facts imply that under the above operations with P the flow ϕ (resp. ϕ') is transformed into a set of pairwise disjoint directed paths (a flow) going from $R_{I \triangle (\pi \cap Y^{\mathrm{r}})}$ to $C_{J \triangle (\pi \cap Y^{\mathrm{c}})}$ (resp. from $R_{I' \triangle (\pi \cap Y^{\mathrm{r}})}$ to $C_{J' \triangle (\pi \cap Y^{\mathrm{c}})}$).

Doing so for all $P(\pi)$ with $\pi \in \Pi$, we obtain flows ψ, ψ' from $R_{\widetilde{I}}$ to $C_{\widetilde{J}}$ and from $R_{\widetilde{I}'}$ to $C_{\widetilde{J}'}$, respectively. The equalities in the last sentence of the lemma are easy. ∎

We call the transformation $(\phi, \phi') \overset{\Pi}{\longmapsto} (\psi, \psi')$ in this lemma the *flow exchange operation* for (ϕ, ϕ') using $\Pi \subseteq M(\phi, \phi')$ (or using $\{P(\pi) \colon \pi \in \Pi\}$). Clearly the exchange operation applied to (ψ, ψ') using the same Π returns (ϕ, ϕ'). The picture below illustrates flows ψ, ψ' obtained from ϕ, ϕ' in Fig. 1 by the exchange operations using the single path P_2 (left) and the single path P_3 (right).



So far our description has been close to that given for the commutative case in [5]. From now on we will essentially deal with the quantum version. The next theorem will serve the main working tool in our arguments; its proof appealing to a combinatorial techniques on paths and flows is given in [4].

Theorem 4.4 *Let ϕ be an $(I|J)$-flow, and ϕ' an $(I'|J')$-flow in G. Let (ψ, ψ') be the double flow obtained from (ϕ, ϕ') by the flow exchange operation using a single couple $\pi = \{i, j\} \in M(\phi, \phi')$. Then:*

(i) *when π is an R- or C-couple and $i < j$,*

$$w(\phi)w(\phi') = q w(\psi) w(\psi') \qquad \text{in case} \quad i \in I \cup J;$$
$$w(\phi)w(\phi') = q^{-1} w(\psi) w(\psi') \qquad \text{in case} \quad i \in I' \cup J';$$

(ii) *when π is an RC-couple, $w(\phi)w(\phi') = w(\psi)w(\psi')$.*

An immediate consequence from this theorem is the following

Corollary 4.5 *For an $(I|J)$-flow ϕ and an $(I'|J')$-flow ϕ', let (ψ,ψ') be obtained from (ϕ,ϕ') by the flow exchange operation using a set $\Pi\subseteq M(\phi,\phi')$. Then*

$$w(\phi)w(\phi')=q^{\zeta^\circ-\zeta^\bullet}w(\psi)w(\psi'), \tag{4.5}$$

where $\zeta^\circ=\zeta^\circ(I|J,I'|J';\Pi)$ (resp. $\zeta^\bullet=\zeta^\bullet(I|J,I'|J';\Pi)$) is the number of R- or C-couples $\pi=\{i,j\}\in\Pi$ such that $i<j$ and $i\in I\cup J$ (resp. $i\in I'\cup J'$).

Indeed, the flow exchange operation using the whole Π reduces to performing, step by step, the exchange operations using single couples $\pi\in\Pi$ (taking into account that for any current double flow (η,η') occurring in the process, the sets $E_\eta\cup E_{\eta'}$ and $E_\eta\triangle E_{\eta'}$, as well as the matching $M(\eta,\eta')$, do not change; cf. Lemma 4.3). Then (4.5) follows from Theorem 4.4.

5 Quadratic relations

As before, we consider a grid-shaped graph $G=(V,E;R,C)$ and the weight function w which is initially defined on the edges of G by (2.3) and then extends to paths and flows according to (2.1) and (3.2). This gives rise to the minor-type function f_G on the set $\mathcal{E}^{m,n}=\{(I|J)\colon I\subseteq[m],\ J\in[n],\ |I|=|J|\}$, by (3.3). In this section, based on Corollary 4.5 describing the transformation of the weights of double flows under the exchange operation, and developing a q-version of the flow-matching method elaborated for the commutative case in [5], we establish sufficient conditions of a general form on quantized quadratic relations for f_G to be valid independently of (grid-shaped) G and some other objects, referring to them as "universal quadratic identities".

Relations of our interest are of the form

$$\sum\nolimits_{\mathcal{I}} q^{\alpha(I|J,I'|J')}f(I|J)f(I'|J')=\sum\nolimits_{\mathcal{K}} q^{\beta(K|L,K'|L')}f(K|L)f(K'|L'), \tag{5.1}$$

where α,β are integer-valued, $\mathcal{I}$ is a family of (possibly multiplied) corteges $(I|J,I'|J')\in\mathcal{E}^{m,n}\times\mathcal{E}^{m,n}$, and similarly for $\mathcal{K}$ (then (5.1) is equivalent to (1.2) with δ corresponding to (α,β)). We usually assume that $\mathcal{I}$ and $\mathcal{K}$ are *homogeneous*, in the sense that for any $(I|J,I'|J')\in\mathcal{I}$ and $(K|L,K'|L')\in\mathcal{K}$,

$$I\cup I'=K\cup K',\quad J\cup J'=L\cup L',\quad I\cap I'=K\cap K',\quad J\cap J'=L\cap L'. \tag{5.2}$$

Moreover, we shall see that only the refinements $(I^\circ,I^\bullet,J^\circ,J^\bullet)$ and $(K^\circ,K^\bullet,L^\circ,L^\bullet)$ are important, whereas the sets $I\cap I'$ and $J\cap J'$ are, in fact, indifferent. (As before, I° means $I-I'$, $I^\bullet$ means $I'-I$, and so on.)

To formulate our validity criterion, we need some definitions and notation.

• We say that a tuple $(I|J,I'|J';M)$, where $(I|J,I'|J')\in\mathcal{I}$ and $M\in\mathcal{M}_{I^\circ,I^\bullet,J^\circ,J^\bullet}$ (cf. (4.3)), is a *configuration* for $\mathcal{I}$. The family of all configurations for $\mathcal{I}$ is denoted by $\mathbf{C}(\mathcal{I})$. Similarly, we define the family $\mathbf{C}(\mathcal{K})$ of configurations for $\mathcal{K}$.

• Define $\mathbf{M}(\mathcal{I})$ to be the family of all matchings M occurring in the members of $\mathbf{C}(\mathcal{I})$, respecting multiplicities (i.e., $\mathbf{M}(\mathcal{I})$ may be a multiset). Define $\mathbf{M}(\mathcal{K})$ in a similar way.

Definition. Families $\mathcal{I}$ and $\mathcal{K}$ are called *balanced* (borrowing terminology from [5]) if there exists a bijection $(I|J,I'|J';M) \stackrel{\gamma}{\longmapsto} (K|K',L|L';M')$ between $\mathbf{C}(\mathcal{I})$ and $\mathbf{C}(\mathcal{K})$ such that $M=M'$. In other words, $\mathcal{I}$ and $\mathcal{K}$ are balanced if $\mathbf{M}(\mathcal{I})=\mathbf{M}(\mathcal{K})$.

Definition. We say that families $\mathcal{I}$ and $\mathcal{K}$ along with functions $\alpha:\mathcal{I}\to\mathbb{Z}$ and $\beta:\mathcal{K}\to\mathbb{Z}$ are *q-balanced* if there exists a bijection γ as above such that, for each $(I|J,I'|J';M)\in\mathbf{C}(\mathcal{I})$ and for $(K|K',L|L';M)=\gamma(I|J,I'|J';M)$, there holds

$$\beta(K|K',L|L')-\alpha(I|J,I'|J')=\zeta^\circ-\zeta^\bullet. \tag{5.3}$$

(In particular, $\mathcal{I},\mathcal{K}$ are balanced.) Here $\zeta^\circ,\zeta^\bullet$ are defined according to Corollary 4.5. Namely, $\zeta^\circ=\zeta^\circ(I|J,I'|J';\Pi)$ and $\zeta^\bullet=\zeta^\bullet(I|J,I'|J';\Pi)$, where Π is the set of couples $\pi\in M$ such that the white/black colors of the elements of π in the refined corteges $(I^\circ,I^\bullet,J^\circ,J^\bullet)$ and $(K^\circ,K^\bullet,L^\circ,L^\bullet)$ are different. (Then ζ° ($\zeta^\bullet$) is the number of R- and C-couples $\{i,j\}\in\Pi$ with $i<j$ and $i\in I^\circ\cup J^\circ$ (resp. $i\in I^\bullet\cup J^\bullet$).) We say that $(K^\circ,K^\bullet,L^\circ,L^\bullet)$ is obtained from $(I^\circ,I^\bullet,J^\circ,J^\bullet)$ by the *index exchange operation* using Π, and may write $\zeta^\circ(I^\circ,I^\bullet,J^\circ,J^\bullet;\Pi)$ for ζ°, and $\zeta^\bullet(I^\circ,I^\bullet,J^\circ,J^\bullet;\Pi)$ for $\zeta^\bullet$.

Theorem 5.1 *Let $\mathcal{I}$ and $\mathcal{K}$ be homogeneous families on $\mathcal{E}^{m,n}\times\mathcal{E}^{m,n}$, and let $\alpha:\mathcal{I}\to\mathbb{Z}$ and $\beta:\mathcal{K}\to\mathbb{Z}$. Suppose that $\mathcal{I},\mathcal{K},\alpha,\beta$ are q-balanced. Then for any grid-shaped graph $G=(V,E;R,C)$, relation (5.1) is valid for $f=f_G$.*

Proof It is close to the proof for the commutative case in [5, Proposition 3.2].

For G fixed, we denote by $\mathcal{D}(\widetilde{I}|\widetilde{J},\widetilde{I}'|\widetilde{J}')$ the set of double flows in G for $(\widetilde{I}|\widetilde{J},\widetilde{I}'|\widetilde{J}')\in\mathcal{I}\cup\mathcal{K}$. A summand concerning $(I|J,J'|J')\in\mathcal{I}$ in the L.H.S. of (5.1) can be expressed via double flows as follows, ignoring the factor of $q^{\alpha(\cdot)}$:

$$\begin{aligned}
f(I|J)f(I'|J') &= \Big(\sum\nolimits_{\phi\in\Phi_G(I|J)} w(\phi)\Big)\times\Big(\sum\nolimits_{\phi'\in\Phi_G(I'|J')} w(\phi')\Big)\\
&= \sum\nolimits_{(\phi,\phi')\in\mathcal{D}(I|J,I'|J')} w(\phi)w(\phi')\\
&= \sum\nolimits_{M\in\mathcal{M}_{I^\circ,I^\bullet,J^\circ,J^\bullet}}\sum\nolimits_{(\phi,\phi')\in\mathcal{D}(I|J,I'|J')\,:\,M(\phi,\phi')=M} w(\phi)w(\phi').
\end{aligned} \tag{5.4}$$

The summand for $(K|L,K'|L')\in\mathcal{K}$ in the R.H.S. of (5.1) is expressed similarly.

Consider a configuration $S=(I|J,I'|J';M)\in\mathbf{C}(\mathcal{I})$ and suppose that (ϕ,ϕ') is a double flow for $(I|J,I'|J')$ with $M(\phi,\phi')=M$ (if such a double flow in G exists). Since $\mathcal{I},\mathcal{K},\alpha,\beta$ are q-balanced, S is bijective to some configuration $S'=(K|L,K'|L';M)\in\mathbf{C}(\mathcal{K})$ satisfying (5.3). As explained earlier, the cortege $(K|L,K'|L')$ is obtained from $(I|J,I'|J')$ by the index exchange operation using some $\Pi\subseteq M$. Then the flow exchange operation applying to (ϕ,ϕ') using this Π results in a double flow (ψ,ψ') for $(K|L,K'|L')$ which satisfies relation (4.5) in Corollary 4.5. Comparing (4.5) with (5.3), we observe that

$$q^{\alpha(I|J,I'|J')}w(\phi)w(\phi')=q^{\beta(K|K',L|L')}w(\psi)w(\psi').$$

Furthermore, such a map $(\phi,\phi')\mapsto(\psi,\psi')$ gives a bijection between all double flows concerning configurations in $\mathbf{C}(\mathcal{I})$ and those in $\mathbf{C}(\mathcal{K})$. Now the desired equality (5.1) follows by considering the last term in expression (5.4) and the corresponding term in the analogous expression concerning $\mathcal{K}$. ∎

As a consequence of Theorems 3.3 and 5.1, the following result is obtained.

Corollary 5.2 *If $\mathcal{I},\mathcal{K},\alpha,\beta$ as above are q-balanced, then relation (5.1) is valid for the corresponding function of minors in the algebra $\mathcal{R}$ of quantum $m\times n$ matrices.*

Remark 3. When speaking of a *universal quadratic identity* of the form (5.1) with homogeneous $\mathcal{I}$ and $\mathcal{K}$, abbreviated as a *UQ identity*, we mean that it depends neither on the graph G nor on the field $\mathbb{K}$ and element $q\in\mathbb{K}^*$, and that the index sets can be modified as follows. Given $(I|J,I'|J')\in\mathcal{I}$, let $A:=I\triangle I'$, $B:=J\triangle J'$, $S:=I\cap I'$ and $T:=J\cap J'$ (by the homogeneity, these sets do not depend on $(I|J,I'|J)\in\mathcal{I}\cup\mathcal{K}$). Take arbitrary $\widetilde m\ge|A|$ and $\widetilde n\ge|B|$ and replace A,B,S,T by disjoint sets $\widetilde A,\widetilde S\subseteq[\widetilde m]$ and disjoint sets $\widetilde B,\widetilde T\subseteq[\widetilde n]$ such that

$$|\widetilde A|=|A|,\quad |\widetilde B|=|B|,\quad |\widetilde S|-|\widetilde T|=|S|-|T|.$$

Let $\nu:A\to\widetilde A$ and $\mu:B\to\widetilde B$ be the order preserving maps. Transform each $(I|J,I'|J')\in\mathcal{I}$ into $(\widetilde I|\widetilde J,\widetilde{I'}|\widetilde{J'})$, where

$$\widetilde I:=\widetilde S\cup\nu(I-S),\quad \widetilde{I'}:=\widetilde S\cup\nu(I'-S),\quad \widetilde J:=\widetilde T\cup\mu(J-T),\quad \widetilde{J'}:=\widetilde T\cup\mu(J'-T),$$

forming a new family $\widetilde{\mathcal{I}}$ on $\mathcal{E}^{\widetilde m,\widetilde n}\times\mathcal{E}^{\widetilde m,\widetilde n}$. Transform $\mathcal{K}$ into $\widetilde K$ in a similar way. One can see that if $\mathcal{I},\mathcal{K},\alpha,\beta$ are q-balanced, then so are $\widetilde{\mathcal{I}},\widetilde{\mathcal{K}}$, keeping α,β. Therefore, if (5.1) is valid for $\mathcal{I},\mathcal{K}$, then it is valid for $\widetilde{\mathcal{I}},\widetilde{\mathcal{K}}$ as well.

Thus, the condition of q-balancedness is sufficient for validity of relation (5.1) for minors of any q-matrix. In Section 7 we shall see that this condition is necessary as well (Theorem 7.1).

Identity (5.1), where all summands have positive signs, is regarded as being written in the canonical form. Sometimes, however, it is more convenient to consider equivalent identities having negative summands in one or both sides (e.g. in the form (1.2)). Also one may multiply all summands in the identity by the same degree of q.

Remark 4. A useful fact is that once we are given an instance of (5.1), we can form another identity by changing the white/black coloring in all refined corteges. More precisely, for a cortege $S=(I|J,I'|J')$, let us say that the cortege $S^{\rm rev}:=(I'|J',I|J)$ is *reversed* to S. Given a family $\mathcal{I}$ of corteges, the *reversed* family $\mathcal{I}^{\rm rev}$ is formed by the corteges reversed to those in $\mathcal{I}$. Then the following property takes place.

Proposition 5.3 *Suppose that $\mathcal{I},\mathcal{K},\alpha,\beta$ are q-balanced. Then $\mathcal{I}^{\rm rev},\mathcal{K}^{\rm rev},-\alpha,-\beta$ are q-balanced as well. Therefore (by Theorem 5.1),*

$$\sum_{(I|J,I'|J')\in\mathcal{I}} q^{-\alpha(I|J,I'|J')}f(I'|J')f(I|J)=\sum_{(K|L,K'|L')\in\mathcal{K}} q^{-\beta(K|L,K'|L')}f(K'|L')f(K|L). \quad (5.5)$$

Proof Let γ be a bijection of $\mathbf{C}(\mathcal{I})$ to $\mathbf{C}(\mathcal{K})$ according to the definition of q-balancedness. Then γ induces a bijection of $\mathbf{C}(\mathcal{I}^{\rm rev})$ to $\mathbf{C}(\mathcal{K}^{\rm rev})$ (also denoted as γ). Namely, if $S = (I|J, I'|J') \in \mathcal{I}$ and $T = (K|L, K'|L') \in \mathcal{K}$ are such that $\gamma(S; M) = (T; M)$ (for some matching M), then we define $\gamma(S^{\rm rev}; M) := (T^{\rm rev}; M)$. When coming from S to $S^{\rm rev}$, each R- or C-couple $\{i, j\}$ in M changes the colors of both elements i, j. This leads to swapping ζ° and $\zeta^\bullet$, i.e., we have $\zeta^\circ(S^{\rm rev}; \Pi) = \zeta^\bullet(S; \Pi)$ and $\zeta^\bullet(S^{\rm rev}; \Pi) = \zeta^\circ(S; \Pi)$ (where Π is the submatching in M involved in the exchange operation). Now (5.5) follows from relation (5.3). ∎

Another useful equivalent transformation concerns swapping row and column indices. Namely, for a cortege $S = (I|J, I'|J')$, the *transposed* cortege is $S^\top := (J|I, J'|I')$, and the family $\mathcal{I}^\top$ *transposed* to $\mathcal{I}$ consists of the corteges $S^\top$ for $S \in \mathcal{I}$, and similarly for $\mathcal{K}$. One can see that the corresponding values ζ° and $\zeta^\bullet$ preserve when coming from $\mathcal{I}$ to $\mathcal{I}^\top$ and from $\mathcal{K}$ to $\mathcal{K}^\top$, and therefore (5.3) implies the identity

$$\sum_{(I|J,I'|J')\in\mathcal{I}} q^{\alpha(I|J,I'|J')} f(J|I) f(J'|I') = \sum_{(K|L,K'|L')\in\mathcal{K}} q^{\beta(K|L,K'|L')} f(L|K) f(L'|K'). \quad (5.6)$$

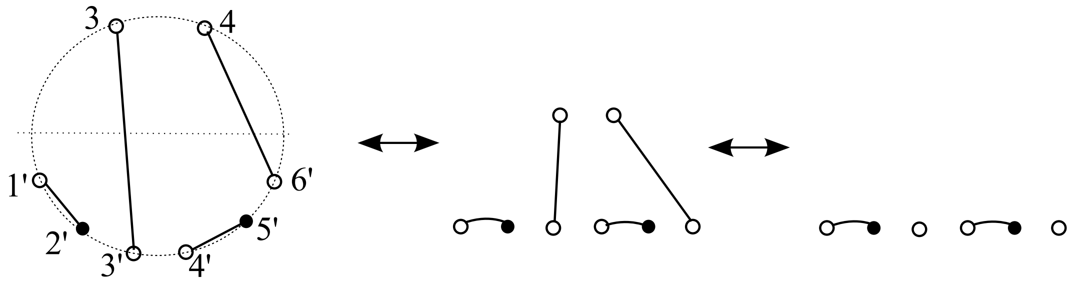
We conclude this section with a simple algorithm which has as the input a corresponding quadruple $\mathcal{I}, \mathcal{K}, \alpha, \beta$ and recognizes the q-balanced for it. Therefore, in light of Theorems 5.1 and 7.1, the algorithm decides whether or not the given quadruple determines a UQ identity of the form (5.1).

Algorithm. Compute the set $\mathcal{M}_{I^\circ, I^\bullet, J^\circ, J^\bullet}$ of feasible matchings M for each $(I|J, I'|J') \in \mathcal{I}$, and similarly for $\mathcal{K}$. For each instance M occurring in these, we extract the family $\mathbf{C}_M(\mathcal{I})$ of all configurations concerning M in $\mathbf{C}(\mathcal{I})$, and extract a similar family $\mathbf{C}_M(\mathcal{K})$ in $\mathbf{C}(\mathcal{K})$. If $|\mathbf{C}_M(\mathcal{I})| \ne |\mathbf{C}_M(\mathcal{K})|$ for at least one instance M, then $\mathcal{I}$ and $\mathcal{K}$ are not balanced at all. Otherwise for each M, we seek for a required bijection $\gamma_M : \mathbf{C}_M(\mathcal{I}) \to \mathbf{C}_M(\mathcal{K})$ by solving the maximum matching problem in the corresponding bipartite graph H_M. More precisely, the vertices of H_M are the tuples $(I|J, I'|J'; M)$ and $(K|L, K'|L'; M)$ occurring in $\mathbf{C}_M(\mathcal{I})$ and $\mathbf{C}_M(\mathcal{K})$, and such tuples are connected by edge in H_M if they obey (5.3). Find a maximum matching N in H_M. (There are many fast algorithms to solve this classical problem; for a survey, see, e.g. [16].) If $|N| = |\mathbf{C}_M(\mathcal{I})|$, then N determines the desired γ_M in a natural way. Taking together, these γ_M give a bijection between $\mathbf{C}(\mathcal{I})$ and $\mathbf{C}(\mathcal{K})$ as required, implying that $\mathcal{I}, \mathcal{K}, \alpha, \beta$ are q-balanced And if $|N| < |\mathbf{C}_M(\mathcal{I})|$ for at least one instance M, then the algorithm declares the non-q-balancedness.

6 Examples of universal quadratic identities

The flow-matching method described above is well adjusted to prove, relatively easily, classical or less known quadratic identities. In this section we give a number of appealing illustrations.

Instead of circular diagrams as in Section 4, we will use more compact, but equivalent, *two-level diagrams*. Also when dealing with a flag pair $I|J$, i.e., when I consists of the elements $1, 2, \dots, |I|$, we may use an appropriate *one-level diagrams*, which leads to no loss of generality. For example, the refined cortege ($I^\circ = \{3, 4\}$, $I^\bullet = \emptyset$, $J^\circ = \{1', 3', 4', 6'\}$, $J^\bullet = \{2', 5'\}$) with the feasible matching $\{1'2', 4'5', 33', 46'\}$ can be visualized in three possible ways as:



A couple $\{i, j\}$ may be denoted as ij. Also for brevity we write $Xi \dots j$ for $X \cup \{i, \dots, j\}$, where X and $\{i, \dots, j\}$ are disjoint.

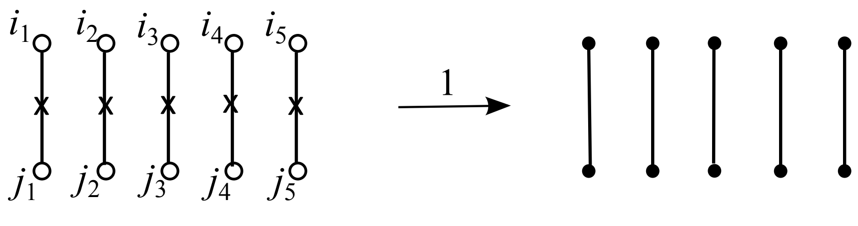
For a value of the flow-generated function $f = f_G$ on $(I|J) \in \mathcal{E}^{m,n}$, we also may use notation $[I|J]$, as though thinking of f as the function of minors of the path matrix Path_G (defined in Section 3). In the flag case $[I|J|$ is usually abbreviated to $[J]$ (in view of $I = \{1, \dots, |J|\}$).

6.1 Commuting minors. We start with a simple illustration of our method by showing that minors $[I|J]$ and $[I'|J']$ (concerning a flow-generated function f_G) "purely" commute when $I' \subset I$ and $J' \subset J$. (This matches the known fact that a minor of a q-matrix commutes with any of its subminors, or that the q-determinant of a (square) q-matrix is a central element of the corresponding algebra.)

Let $I^\circ = I - I'$ consist of $i_1 < \dots < i_k$, and $J^\circ = J - J'$ consist of $j_1 < \dots < j_k$. Since $I^\bullet = I' - I = \emptyset$ and $J^\bullet = J' - J = \emptyset$, there is only one feasible matching M for $(I^\circ, I^\bullet, J^\circ, J^\bullet)$; namely, the one formed by the RC-couples $\pi_\ell = i_\ell j_\ell$, $\ell = 1, \dots, k$. The index exchange operation applied to $(I|J, I'|J')$ using the whole M produces the cortege $(K|L, K'|L')$ for which $K^\circ = I^\bullet = \emptyset$, $K^\bullet = I^\circ$, $L^\circ = J^\bullet = \emptyset$, $L^\bullet = J^\circ$ (and $K \cap K' = I \cap I'$, $L \cap L' = J \cap J'$). Since M consists of only RC-couples, we have $\zeta^\circ(I^\circ, I^\bullet, J^\circ, J^\bullet; M) = \zeta^\bullet(I^\circ, I^\bullet, J^\circ, J^\bullet; M) = 0$. So the (one-element) families $\mathcal{I} = \{(I|J, I'|J')\}$ and $\mathcal{K} = \{(K|L, K'|L')\}$ along with $\alpha = \beta = 0$ are q-balanced, and Theorem 5.1 gives the desired equality

$$[I|J][I'|J'] = [I'|J'][I|J].$$

This is illustrated in the picture with two-level diagrams (in case $k = 5$). Hereinafter we indicate by crosses the couples that are involved in the index exchange operation that is applied (i.e., the couples where the colors of elements are changed).



6.2 Quasicommuting minors. Recall that two sets $I, J \subseteq [n]$ are called *weakly separated* if, up to renaming I and J, there holds: $|I| \ge |J|$, and $J - I$ has a partition $J_1 \cup J_2$ such that $J_1 < I - J < J_2$ (where we write $X < Y$ if $x < y$ for any $x \in X$ and $y \in Y$). Leclerc and Zelevinsky proved the following

Theorem 6.1 ([12]) *Two flag minors $[I]$ and $[J]$ of a quantum matrix* quasicommute*, i.e., satisfy*

$$[I][J] = q^c[J][I] \tag{6.1}$$

for some $c \in \mathbb{Z}$, if and only if the column sets I, J are weakly separated. Moreover, when $|I| \ge |J|$ and $J_1 \cup J_2$ is a partition of $J - I$ with $J_1 < I - J < J_2$, the number c in (6.1) is equal to $|J_2| - |J_1|$.

(In case $I \cap J = \emptyset$, "if" part is due to Krob and Leclerc [10]). We explain how to obtain "if" part of Leclerc–Zelevinsky's theorem by use of the flow-matching method.

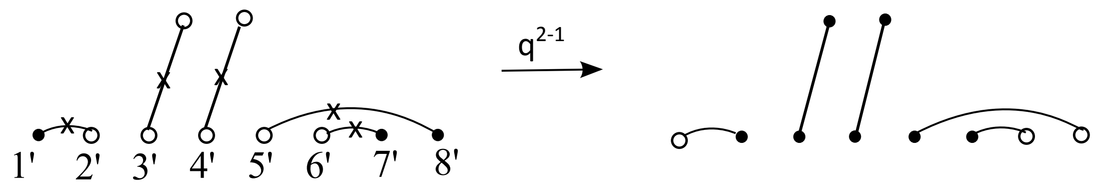
Let $A := \{1, \dots, |I|\}$, $B := \{1, \dots, |J|\}$, and define

$$A^\circ := A - B, \quad B^\bullet := B - A \ (= \emptyset), \quad I^\circ := I - J, \quad J^\bullet := J - I.$$

One can see that $(A^\circ, B^\bullet, I^\circ, J^\bullet)$ has exactly one feasible matching M; namely, J_1 is coupled with the first $|J_1|$ elements of I°, J_2 is coupled with the last $|J_2|$ elements of I° (forming all C-couples), and the rest of I° is coupled with A° (forming all RC-couples).

Observe that the index exchange operation applied to $(A|I, B|J)$ using the whole M swaps $A|I$ and $B|J$ (since it produces the cortege $(K|L, K'|L')$ such that $K^\circ = B^\bullet = \emptyset$, $K^\bullet = A^\circ$, $L^\circ = J^\bullet$, $L^\bullet = I^\circ$) Also M consists of $|J_1| + |J_2|$ C-couples and $|A^\circ|$ RC-couples. Moreover, the C-couples are partitioned into $|J_1|$ couples ij with $i < j$ and $i \in J_1$, and $|J_2|$ couples ij with $i < j$ and $j \in J_2$. This gives $\zeta^\circ = |J_2|$ and $\zeta^\bullet = |J_1|$. Hence the (one-element) families $\{(A|I, B|J)\}$ and $\{(B|J, A|I)\}$ along with $\alpha(A|I, B|J) = 0$ and $\beta(B|J, A|I) = |J_2| - |J_1|$ are q-balanced. Now Theorem 5.1 implies (6.1) with $c = |J_2| - |J_1|$.

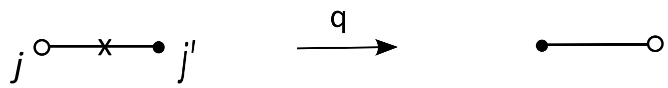
The picture with two-level diagrams illustrates the case $|I - J| = 5$, $|J - I| = 3$, $|J_1| = 1$ and $|J_2| = 2$.



"Only if" part of Theorem 6.1 will be discussed in Section 8. Also we will give there a generalization of this theorem that characterizes the set of all pairs of quasicommuting minors (not necessarily flag ones).

6.3 Quantum relations in path matrices. Next we prove Theorem 3.2 on q-relations for entries of the path matrix Path_G of a grid-shaped graph G.

(a) To see the assertion for a 1×2 submatrix $(a\ b)$ of Path_G, let $a=[i|j]$ and $b=[i|j']$ (then $j<j'$). The cortege $S=(i|j,i|j')$ admits a unique feasible matching; it consists of the single C-couple $\pi=jj'$. The index exchange operation using π transforms S into $T=(i|j',i|j)$; see the picture with one-level diagrams:

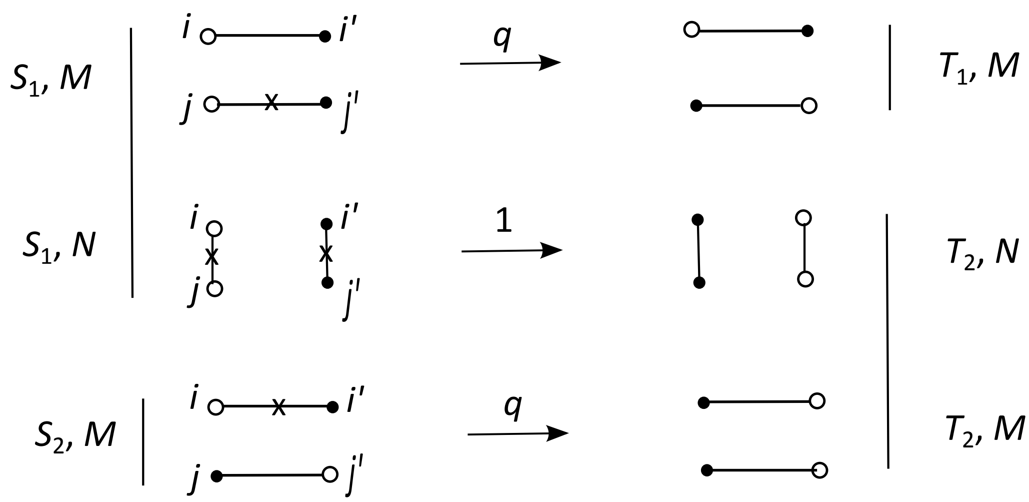


We observe that $\{S\}$ and $\{T\}$ along with $\alpha=0$ and $\beta=1$ $(=\zeta^\circ-\zeta^\bullet)$ are q-balanced, and Theorem 5.1 yields $ab=qba$, as required.

(b) For a 2×1 submatrix $\binom{c}{a}$ of Path_G, the argument is similar (with the difference that, for $a=[i|j]$ and $c=[i'|j]$ $(i<i')$, the unique feasible matching consists of the single R-couple $\pi=ii'$).

(c) Consider a 2×2 submatrix $\binom{c\ d}{a\ b}$ of Path_G, where $a=[i|j]$, $b=[i|j']$, $c=[i'|j]$, $d=[i'|j']$ (then $i<i'$ and $j<j'$). Let $\mathcal{I}$ consist of two corteges $S_1=(i|j,i'|j')$, $S_2=(i|j',i'|j)$, and $\mathcal{K}$ consist of two corteges $T_1=(i|j',i'|j)$, $T_2=(i'|j',i|j)$ (note that $S_2=T_1$).

Observe that S_1 admits 2 feasible matchings, namely, $M=\{ii',jj'\}$ and $N=\{ij,i'j'\}$, while S_2 admits only one feasible matching M. In their turn, $\mathcal{M}(T_1)=\{M\}$ and $\mathcal{M}(T_2)=\{M,N\}$. Hence we can form the bijection between $\mathbf{C}(\mathcal{I})$ and $\mathbf{C}(\mathcal{K})$ that sends the configuration $(S_1;M)$ to $(T_1;M)$, (S_1,N) to $(T_2;N)$, and (S_2,M) to $(T_2;M)$. This bijection is illustrated in the picture (where, as before, we indicate the submathings involved in the exchange operations with crosses).



Assign

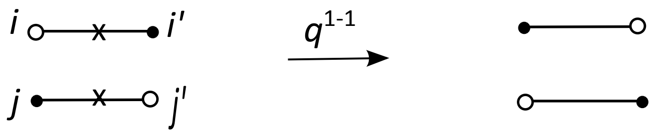
$$\alpha(S_1)=0,\quad \alpha(S_2)=-1,\quad \beta(T_1)=1,\quad \beta(T_2)=0.$$

One can observe from the above diagrams that $\mathcal{I},\mathcal{K},\alpha,\beta$ are q-balanced. We obtain

$$[i|j][i'|j']+q^{-1}[i|j'][i'|j]=q[i|j'][i'|j]+[i'|j'][i|j],$$

yielding $ad-da=(q-q^{-1})bc$, as required.

Finally, to see $bc = cb$, take the 1-element families $\{S' = (i|j', i'|j)\}$ and $\{T' = (i'|j, i|j')\}$; then $\{ii', jj'\}$ is the only feasible matching for each of S', T'. The above families along with $\alpha = \beta = 0$ are q-balanced, as is seen from the picture:



This gives $[i|j'][i'|j] = [i'|j][i|j']$, or $bc = cb$, as required.

6.4 Relations with triples and quadruples. In the commutative case (when dealing with the commutative coordinate ring of $m \times n$ matrices over a field), the simplest examples of quadratic identities on flag minors are presented by the classical Plücker relations involving 3- and 4-element sets of columns. More precisely, for $A \subseteq [n]$, let $g(A)$ denote the flag minor with the set A of columns. Then for any three elements $i < j < k$ in $[n]$ and a set $X \subseteq [n] - \{i, j, k\}$, there holds

$$g(Xik)g(Xj) = g(Xij)g(Xk) + g(Xjk)g(Xi), \tag{6.2}$$

and for any $i < j < k < \ell$ and $X \subseteq [n] - \{i, j, k, \ell\}$,

$$g(Xik)g(Xj\ell) = g(Xij)g(Xk\ell) + g(Xj\ell)g(Xjk), \tag{6.3}$$

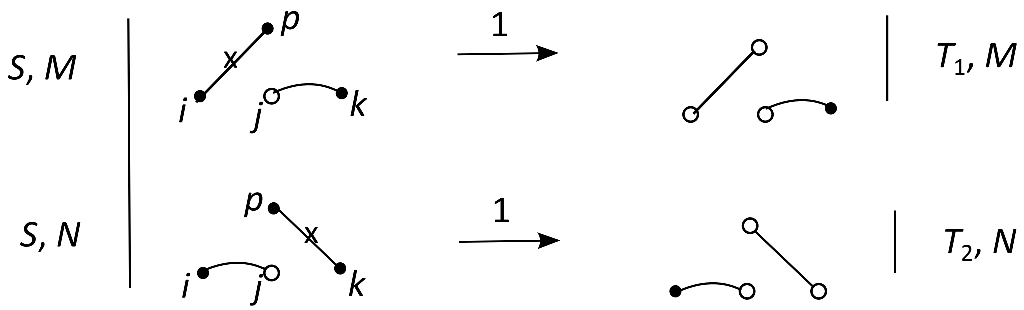
There are two quantized counterparts of (6.2) (concerning flag minors of the matrix Path_G). One of them is viewed as

$$[Xj][Xik] = [Xij][Xk] + [Xjk][Xi], \tag{6.4}$$

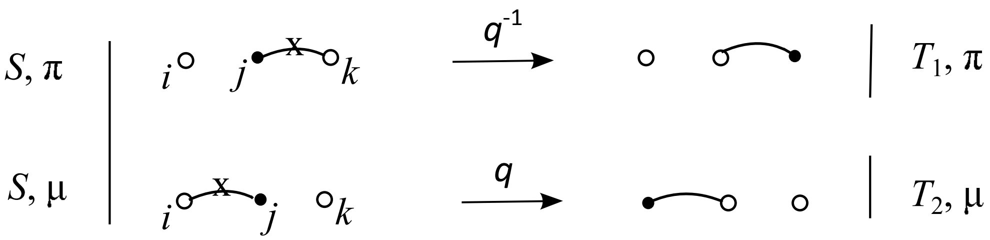
and the other as

$$[Xik][Xj] = q^{-1}[Xij][Xk] + q[Xjk][Xi]. \tag{6.5}$$

To see (6.4), associate to Xj the white pair $(I^\circ, J^\circ) = (\emptyset|\{j\})$, and to Xik the black pair $(I^\bullet|J^\bullet) = (\{p\}|\{i, k\})$, where p is the last row index for $[Xik]$ (i.e., $p = |X| + 2$). Then $\mathcal{M}_{I^\circ, I^\bullet, J^\circ, J^\bullet}$ consists of two feasible matchings: $M = \{pi, jk\}$ and $N = \{ij, pk\}$. Now (6.4) is seen by considering the following picture with two-level diagrams, where we write S for the cortege $([p-1]\,|Xj, [p]\,|Xik)$, T_1 for $([p]\,|Xij, [p-1]\,|Xk)$, and T_2 for $([p]\,|Xjk, [p-1]\,|Xi)$:



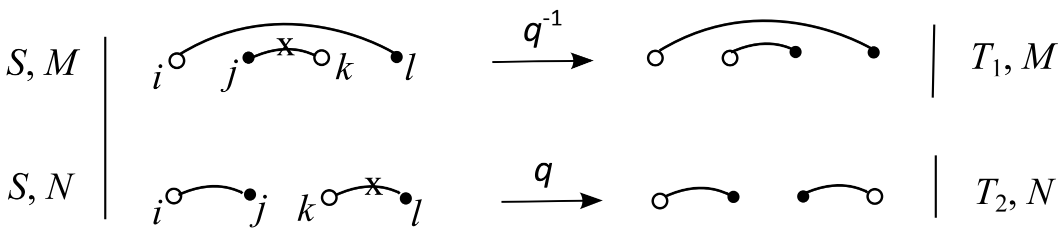
As to (6.5), it suffices to consider one-level diagrams (as we are not going to use RC-couples in the exchange operations). Now the "white" object is the column set $J^\circ = \{i,k\}$ and the "black" object is $J^\bullet = \{j\}$. Then $\mathcal{M}_{\{p\},\emptyset,J^\circ,J^\bullet}$ consists of two feasible matchings, one using the C-couple $\pi = jk$, and the other using the C-couple $\mu = ij$. Now (6.5) can be seen from the picture, where S stands for (Xik, Xj), T_1 for (Xij, Xk), and T_2 for (Xjk, Xi).



Next we demonstrate the following quantized counterpart of (6.3):

$$[Xik][Xj\ell] = q^{-1}[Xij][Xk\ell] + q[Xi\ell][Xjk]. \tag{6.6}$$

To see this, we use one-level diagrams and consider the column sets $J^\circ = \{i,k\}$ and $J^\bullet = \{j,\ell\}$. Then $\mathcal{M}_{\emptyset,\emptyset,J^\circ,J^\bullet}$ consists of two feasible matchings: $M = \{i\ell, jk\}$ and $N = \{ij, k\ell\}$. Identity (6.6) can be observed from the picture, where $S = (Xik, Xj\ell)$, $T_1 = (Xij, Xk\ell)$ and $T_2 = (Xi\ell, Xjk)$.



Remark 5. Note that, if one wishes, one can produce more identities from (6.4) and (6.5), using the fact that Xij and Xk (as well as Xjk and Xi) are weakly separated, and therefore their corresponding flag minors quasicommute (see Section 6.2). In contrast, Xj and Xik are not weakly separated. Next, subtracting from (6.5) identity (6.4) multiplied by q results in the identity of the form

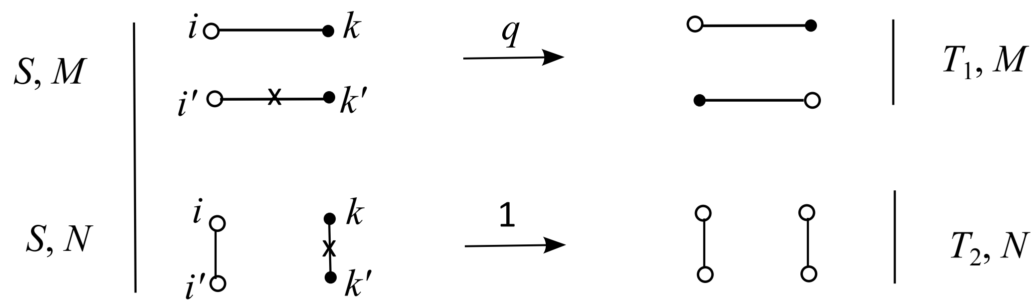
$$[Xik][Xj] = q[Xj][Xik] - (q - q^{-1})[Xij][Xk],$$

which is in spirit of *commutation relations* for quantum minors studied in [8, 9].

6.5 Dodgson's type identity. As one more simple illustration of our method, we consider a q-analogue of the classical Dodgson's condensation formula for usual minors [6]. It can be stated as follows: for elements $i < k$ of $[m]$, a set $X \subseteq [m] - \{i,k\}$, elements $i' < k'$ of $[n]$, and a set $X' \subseteq [n] - \{i',k'\}$ (with $|X'| = |X|$),

$$[Xi|X'i'][Xk|X'k'] = q[Xi|X'k'][Xk|X'i'] + [Xik|X'i'k'][X|X']. \tag{6.7}$$

In this case we deal with the cortege $S=(I|J,I'|J')=(Xi|X'i',Xk|X'k')$ and its refinement $(I^\circ,I^\bullet,J^\circ,J^\bullet)$ of the form (i,k,i',k'). The latter admits two feasible matchings: $M=\{ik,i'k'\}$ and $N=\{ii',kk'\}$. Now (6.7) can be concluded by examining the picture below, where T_1 stands for the cortege $(Xi|X'k',Xk|X'i')$, and T_2 for $(Xik|X'i'k',X|X')$:



6.6 Two general quadratic identities. Two representable quadratic identities of a general form were established for quantum flag minors in [11, 15].

The first one considers column subsets $I,J\subset[n]$ with $|I|\le|J|$ and is viewed as

$$[I][J]=\sum_{\mu\subseteq J-I,\,|\mu|=|J|-|I|}(-q)^{Inv(J-\mu,\mu)-Inv(I,\mu)}[I\cup\mu][J-\mu], \tag{6.8}$$

where $Inv(A,B)$ denotes the number of pairs $(a,b)\in A\times B$ with $a>b$. Observe that (6.4) is a special case of (6.8) in which the roles of I and J are played by Xj and Xik, respectively. Indeed, in this case μ ranges over the singletons $\{i\}$ and $\{k\}$, and we have $Inv(Xk,i)-Inv(Xj,i)=0$ and $Inv(Xi,k)-Inv(Xj,k)=0$. (For brevity, we write $Inv(\cdot,i')$ for $Inv(\cdot,\{i'\})$.)

The second one considers $I,J\subset[n]$ with $|I|-|J|\ge 2$ and is viewed as

$$\sum\nolimits_{a\in I-J}(-q)^{Inv(a,I-a)-Inv(a,J)}[Ja][I-a]=0 \tag{6.9}$$

(where we write Ja for $J\cup\{a\}$, and $I-a$ for $I-\{a\}$). A special case is (6.6) (with $I=Xjk\ell$ and $J=Xi$).

We explain how (6.8) and (6.9) can be proved for flag minors of Path_G by use of our flow-matching method.

Proof of (6.8). The pair (I,J) corresponds to the cortege $S:=([p]\,|I,\ [p+k]\,|J)$ and its refinement $R:=(\emptyset,\ Q:=\{p+1,\dots,p+k\},\ I^\circ:=I-J,\ J^\bullet:=J-I)$, where $p:=|I|$ and $k:=|J|-|I|$. In its turn, each pair $(I\cup\mu)|(J-\mu)$ occurring in the R.H.S. of (6.8) corresponds to the cortege $S_\mu:=([p+k]\,|(I_\mu:=I\cup\mu),\ [p]\,|(J_\mu:=J-\mu))$ and its refinement $R_\mu:=(Q,\ \emptyset,\ I^\circ_\mu:=I^\circ\cup\mu,\ J^\bullet_\mu:=J^\bullet-\mu)$.

So we deal with the set

$$\mathcal{F}:=\{S\}\cup\{S_\mu\colon \mu\subset J^\bullet,\ |\mu|=k\},$$

of corteges and the related set $\mathbf{C}(\mathcal{F})$ of configurations (of the form $(S;M)$ or $(S_\mu;M)$), and our aim is to construct an involution $\gamma:\mathbf{C}(\mathcal{F})\to\mathbf{C}(\mathcal{F})$ which is agreeable with matchings, signs and q-factors figured in (6.8). (Under reducing (6.8) to the canonical form, $\mathcal{F}$ splits into two families $\mathcal{I}$ and $\mathcal{K}$, and γ determines the q-balancedness for $\mathcal{I},\mathcal{K}$ with corresponding α,β.)

Consider a refined cortege $R_\mu=(Q,\emptyset,I^\circ_\mu,J^\bullet_\mu)$ and a feasible matching M for it. Note that M consists of $k=|Q|$ RC-couples (connecting Q and I°_μ) and $|J^\bullet_\mu|=|I^\circ|$ C-couples (connecting I°_μ and $J^\bullet_\mu$). Two cases are possible.

<u>Case 1</u>: each C-couple connects $J^\bullet_\mu$ and I°. Then all RC-couples in M connect Q and μ. Therefore, the exchange operation applied to S_μ using the set Π of RC-couples of M produces the "initial" cortege S (corresponding to the refinement $R=(\emptyset,Q,I^\circ,J^\bullet)$). Clearly M is a feasible matching for S and the exchange operation applied to S using Π returns S_μ. We link $(S;M)$ and $(S_\mu;M)$ by γ.

Note that for each C-couple $\pi=ij\in M-\Pi$ and each element $q\in\mu$, either $q<i,j$ or $q>i,j$ (otherwise the RC-couple using q would "cross" π, contrary to the planarity requirement (4.3)(ii) for M). This implies $Inv(J_\mu,\mu)=Inv(I_\mu,\mu)$, whence the terms $[I][J]$ in the L.H.S. and $(-q)^0[I_\mu][J_\mu]$ in the R.H.S. of (6.8) are q-balanced.

<u>Case 2</u>: there is a C-couple in M connecting $J^\bullet_\mu$ and μ. Among such couples, choose the couple $\pi=ij$ $(i<j)$ such that: (a) $j-i$ is minimum, and (b) i is minimum subject to (a). From (4.3) and (a) it follows that

(6.10) if a couple $\pi'\in M$ has an element (strictly) between i and j, then π' connects I° and $J^\bullet_\mu$, and the other element of π' is between i and j as well.

Let $S_{\mu'}$ be obtained by applying to S_μ the exchange operation using the single couple π; then $\mu'=\mu\triangle\pi$, $I^\circ_{\mu'}=I^\circ_\mu\triangle\pi$ and $J^\bullet_{\mu'}=J^\bullet_\mu\triangle\pi$. The matching M is feasible for $S_{\mu'}$, we are in Case 2 with $S_{\mu'}$ and M, and one can see that the couple $\pi'\in M$ chosen for $S_{\mu'}$ according to the above rules (a),(b) coincides with π. Based on this, we link $(S_\mu;M)$ and $(S_{\mu'};M)$ by γ.

Now we compute and compare the numbers $a:=Inv(J^\bullet_{\mu'}=J-\mu',\,\mu')-Inv(J^\bullet{}_\mu=J-\mu,\,\mu)$ and $b:=Inv(I,\mu')-Inv(I,\mu)$. Let d be the number of elements of I° between i and j (recall that $\pi=ij$ and $i<j$). Property (6.10) shows that the number of elements of $J^\bullet_\mu$ (as well as of $J^\bullet_{\mu'}$) between i and j is equal to d too. Coonsider two possibilities.

<u>Subcase 2a</u>: $i\in\mu$ (and $j\in J^\bullet_\mu$). Then $i\in J^\bullet_{\mu'}$ and $j\in\mu'$. This implies that $a=Inv(J^\bullet_{\mu'},j)-Inv(J^\bullet_\mu,i)=d+1$ and $b=Inv(I^\circ,j)-Inv(I^\circ,i)=d$.

<u>Subcase 2b</u>: $i\in J^\bullet_\mu$ (and $j\in\mu$). Then $i\in\mu'$ and $j\in J^\bullet_{\mu'}$, yielding $a=-d-1$ and $b=-d$.

Finally, let $(-q)^\alpha$ and $(-q)^\beta$ be the coefficients to the products $[I_\mu][J_\mu]$ and $[I_{\mu'}][J_{\mu'}]$ in (6.8), respectively. Then $\beta-\alpha=a-b$, which is equal to 1 in Subcase 2a and -1 in Subcase 2b. In both cases this amounts to the value $\zeta^\circ-\zeta^\bullet$ for the exchange operation applied to S_μ using π, and validity of (6.8) follows. ∎

Remark 6. Sometimes it is useful to consider the identity formed by the corteges reversed to those in (6.8) (see Remark 4 in Section 5), which is viewed as

$$[J][I] = \sum_{\mu \subseteq J-I,\, |\mu|=|J|-|I|} (-q)^{Inv(I,\mu)-Inv(J-\mu,\mu)}[J-\mu][I\cup\mu].$$

Proof of (6.9). Let $p := |J|$, $k := |I| - |J|$, $Q := [p+k-1] - [p+1]$, $J^\circ := J - I$ and $I^\bullet := I - J$. For $a \in I^\bullet$, the pair $(Ja, I-a)$ in (6.9) corresponds to the cortege $S_a := ([p+1]\,|Ja,\ [p+k-1]\,|(I-a))$ and its refinement $R_a := (\emptyset, Q,\ J^\circ a,\ I^\bullet_a := I^\bullet - a)$ (we use the fact that $k \ge 2$).

We deal with the set $\mathcal{F} := \{S_a \colon a \in I^\bullet\}$ of corteges and the related set $\mathbf{C}(\mathcal{F})$ of configurations $(S_a; M)$, and similar to the previous proof, our aim is to construct an appropriate involution $\gamma : \mathbf{C}(\mathcal{F}) \to \mathbf{C}(\mathcal{F})$.

Consider a refined cortege $R_a = (\emptyset, Q, J^\circ a, I^\bullet_a)$ and a feasible matching M for it. Take the couple in M containing a, say, $\pi = \{a, b\}$. Note that π is a C-couple and $b \in I^\bullet_a$ (taking into account that a is white and $Q, I^\bullet_a$ are black). The exchange operation applied to S_a using π produces the member S_b of $\mathcal{F}$, and we link S_a and S_b by γ.

It remains to estimate the coefficients $(-q)^\alpha$ and $(-q)^\beta$ to the products $[Ja][I-a]$ and $[Jb][I-b]$ in (6.9), respectively.

Let d be the number of elements of $I^\bullet$ between a and b. It is equal to the number of elements of J° between a and b (since, in view of (4.3), the elements of $I^\bullet \cup J^\circ$ between a and b must be partitioned into C-couples in M). This implies that if $a < b$, then $Inv(b, I-b) - Inv(a, I-a) = d+1$ and $Inv(b,J) - Inv(a,J) = d$. Therefore, $\beta - \alpha = (d+1) - d = 1$. And if $a > b$, then $Inv(b, I-b) - Inv(a, I-a) = -d-1$ and $Inv(b,J) - Inv(a,J) = -d$, whence $\beta - \alpha = -1$. In both cases, $\beta - \alpha$ coincides with the corresponding value of $\zeta^\circ - \zeta^\bullet$, and the result follows. ▮

7 Necessity of the q-balancedness

In this section we show a converse assertion to Theorem 5.1, thus obtaining a complete characterization for the UQ identities on quantized minors. This characterization, given in terms of the q-balancedness, justifies the algorithm of recognizing UQ identities described in the end of Section 5. As before, we deal with homogeneous families of corteges in $\mathcal{E}^{m,n} \times \mathcal{E}^{m,n}$.

Theorem 7.1 *Let $\mathbb{K}$ be a field of characteristic zero and let $q \in \mathbb{K}^*$ be transcendental over $\mathbb{Q}$. Suppose that $\mathcal{I}, \mathcal{K}, \alpha, \beta$ (as in Section 5) are not q-balanced. Then there exists (and can be explicitly constructed) a grid-shaped graph G for which relation (5.1) is violated.*

Proof We essentially use an idea and construction worked out for the commutative version in [5, Sec. 5].

Recall that the homogeneity of $\mathcal{F} := \mathcal{I} \sqcup \mathcal{K}$ means that there are invariant sets $X^{\rm r}, Y^{\rm r} \subseteq [m]$ and $X^{\rm c}, Y^{\rm c} \subseteq [n]$ such that

$$I \cap I' = X^{\rm r}, \quad I \triangle I' = Y^{\rm r}, \quad J \cap J' = X^{\rm c}, \quad J \triangle J' = Y^{\rm c} \tag{7.1}$$

hold for any cortege $(I|J, I'|J') \in \mathcal{F}$ (cf. (5.2)).

For a perfect matching M on $Y^{\rm r} \sqcup Y^{\rm c}$, let us denote by $\mathcal{I}_M$ the set of corteges $S = (I|J, I'|J') \in \mathcal{I}$ for which M is feasible (see (4.3)), and denote by $\mathcal{K}_M$ a similar set for $\mathcal{K}$. The q-balancedness of $\mathcal{I}, \mathcal{K}, \alpha, \beta$ would mean that, for any $M \in \mathbf{M}(\mathcal{F})$, there exists a bijection $\gamma_M : \mathcal{I}_M \to \mathcal{K}_M$ respecting (5.3). That is, for any $S = (I|J, I'|J') \in \mathcal{I}_M$ and for $T = (K|L, K'|L') = \gamma_M(S)$, there holds

$$\beta(T) - \alpha(S) = \zeta^\circ(\Pi_{S,T}) - \zeta^\bullet(\Pi_{S,T}). \tag{7.2}$$

Here: $\Pi = \Pi_{S,T}$ is the set of couples $\pi \in M$ where the white/black colors of π in the refined corteges $(I^\circ, I^\bullet, J^\circ, J^\bullet)$ and $(K^\circ, K^\bullet, L^\circ, L^\bullet)$ are different (i.e., the latter is obtained from the former by the index exchange operation using Π), and $\zeta^\circ(\Pi)$ (resp. $\zeta^\bullet(\Pi)$) is the number of R- and C-couples $\{i,j\} \in \Pi$ with $i < j$ and $i \in I^\circ \cup J^\circ$ (resp. $i \in I^\bullet \cup J^\bullet$).

The following relation will be crucial in our argument.

Proposition 7.2 *Let M be a perfect planar matching on $Y^{\rm r} \sqcup Y^{\rm c}$. Then there exists (and can be explicitly constructed) a grid-wise graph $G = (V, E)$ with the following properties: for each cortege $S = (I|J, I'|J') \in \mathcal{E}^{m,n} \times \mathcal{E}^{m,n}$ satisfying (7.1),*

(P1) *if M is feasible for S, then G has a unique $(I|J)$-flow and a unique $(I'|J')$-flow;*

(P2) *if M is not feasible for S, then at least one of $\Phi_G(I|J)$ and $\Phi_G(I'|J')$ is empty.*

We will prove this proposition later, and now, in the assumption of its validity, we complete the proof of the theorem.

Let $\mathcal{I}, \mathcal{K}, \alpha, \beta$ be not q-balanced. Then there exists a matching $M \in \mathbf{M}(\mathcal{F})$ that admits no bijection γ_M as above between $\mathcal{I}_M$ and $\mathcal{K}_M$ (in particular, at least one of $\mathcal{I}_M$ and $\mathcal{K}_M$ is nonempty). We fix such an M and consider a graph G as in Proposition 7.2 for this M.

Our aim is to show that relation (5.1) is violated for the function $f = f_G$ (yielding the theorem). Suppose that this is not so, and (5.1) is valid for f. By (P2) in Proposition 7.2 (and by the definition of f in (3.3)), we have $f(I|J)f(I'|J') = 0$ for each cortege $(I|J, I'|J') \in \mathcal{F} - \mathcal{F}_M$, denoting $\mathcal{F}_M := \mathcal{I}_M \sqcup \mathcal{K}_M$. On the other hand, (P1) in this proposition implies that if $(I|J, I'|J') \in \mathcal{F}_M$, then

$$f(I|J)f(I'|J') = w(\phi_{I|J})\, w(\phi_{I'|J'}),$$

where $\phi_{I|J}$ (resp. $\phi_{I'|J'}$) is the only $(I|J)$-flow (resp. $(I'|J')$-flow) in G. Thus, (5.1) can be rewritten as

$$\sum\nolimits_{\mathcal{I}_M} q^{\alpha(I|J,I'|J')} w(\phi_{I|J}) w(\phi_{I'|J'}) = \sum\nolimits_{\mathcal{K}_M} q^{\beta(K|L,K'|L')} w(\phi_{K|L}) w(\phi_{K'|L'}). \tag{7.3}$$

For each cortege $S = (I|J, I'|J') \in \mathcal{F}_M$, the weight $Q(S) := w(\phi_{I|J})\, w(\phi_{I'|J'})$ of the double flow $(\phi_{I|J}, \phi_{I'|J'})$ is a monomial in edges e of G, regarded as "generators" (or a Laurent monomial in inner vertices of G); see (2.1),(2.3),(3.2). For any two corteges in $\mathcal{F}_M$, one is obtained from the other by the index exchange operation using a submatching of M, and we know from the description in Section 4 that if one double flow is obtained from another by the flow exchange operation, then the (multi)sets of edges occurring in these double flows are the same (cf. Lemma 4.3).

Thus, the (multi)set of edges occurring in the weight monomial $Q(S)$ is the same for all corteges S in $\mathcal{F}_M$. Let us fix an arbitrary linear order σ on E. Then the monomial $Q_\sigma = Q_\sigma(S)$ obtained from $Q(S)$ by a permutation of the entries so as to make them weakly decreasing w.r.t. σ from left to right is the same for all $S \in \mathcal{F}_M$. Therefore, applying the quasicommutation relations on vertices of G, we observe that for $S \in \mathcal{F}_M$, the weight $Q(S)$ is expressed as

$$Q(S) = q^{\rho(S)} Q_\sigma \tag{7.4}$$

for some $\rho(S) \in \mathbb{Z}$. Using such expressions, we rewrite (7.3) as

$$\sum\nolimits_{S\in\mathcal{I}_M} q^{\alpha(S)+\rho(S)} Q_\sigma = \sum\nolimits_{T\in\mathcal{K}_M} q^{\beta(T)+\rho(T)} Q_\sigma,$$

obtaining

$$\sum\nolimits_{S\in\mathcal{I}_M} q^{\alpha(S)+\rho(S)} = \sum\nolimits_{T\in\mathcal{K}_M} q^{\beta(T)+\rho(T)}. \tag{7.5}$$

Since q is transcendental, the polynomials in q in both sides of (7.5) are equal, implying that $|\mathcal{I}_M| = |\mathcal{K}_M|$ and there exists a bijection $\widetilde{\gamma} : \mathcal{I}_M \to \mathcal{K}_M$ such that

$$\alpha(S) + \rho(S) = \beta(\widetilde{\gamma}(S)) + \rho(\widetilde{\gamma}(S)) \qquad \text{for each } S \in \mathcal{I}_M. \tag{7.6}$$

This together with relations of the form (7.4) gives

$$q^{\alpha(S)} Q(S) = q^{\beta(\widetilde{\gamma}(S))} Q(\widetilde{\gamma}(S)),$$

Now, for $S = (I|J, I'|J') \in \mathcal{I}_M$, let $T = (K|L, K'|L') := \widetilde{\gamma}(S)$ and $\Pi := \Pi_{S,T}$. Using relation (4.5) from Corollary 4.5, we have

$$\begin{aligned} q^{\beta(T)-\alpha(S)} Q(T) = Q(S) = w(\phi_{I|J})\, w(\phi_{I'|J'}) \\ = q^{\zeta^\circ(\Pi)-\zeta^\bullet(\Pi)} w(\phi_{K|L})\, w(\phi_{K'|L'}) = q^{\zeta^\circ(\Pi)-\zeta^\bullet(\Pi)} Q(T), \end{aligned}$$

whence $\beta(T) - \alpha(S) = \zeta^\circ(\Pi) - \zeta^\bullet(\Pi)$. Thus, the bijection $\gamma_M := \widetilde{\gamma}$ satisfies (7.2), and we conclude that $\mathcal{I}_M, \mathcal{K}_M, \alpha, \beta$ are q-balanced. This contradiction proves the theorem. ∎

Proof of Proposition 7.2. We utilize the construction of a graph with properties (P1) and (P2) from [5]; denote it by $H = (Z, U)$. We first outline essential details of that construction and then explain how to turn H into an equivalent grid-shaped graph G. A series of transformations of H that we apply to obtain G consists of subdividing some edges $e = (u, v)$ (i.e., replacing e by a directed path from u to v) and shifting a set of vertices and edges in the plane (preserving the planar structure of the graph). Such transformations maintain properties (P1) and (P2), whence the result will follow.

Let $Y^{\rm r}\cup X^{\rm r}=\{1,2,\ldots,k\}$ and $Y^{\rm c}\cup X^{\rm c}=\{1',2',\ldots,k'\}$. Denote the sets of R-, C-, and RC-couples in M by $M^{\rm r}$, $M^{\rm c}$, and $M^{\rm rc}$, respectively. An R-couple $\pi=\{i,j\}$ with $i<j$ is denoted by ij, and we denote by $\prec$ the natural partial order on R-couples where $\pi'\prec\pi$ if $\pi'=pr$ is an R-couple with $i<p<r<j$. And similarly for C-couples. When $\pi'\prec\pi$ and there is no π'' between π and π' (i.e., $\pi'\prec\pi''\prec\pi$), we say that π' is an immediate successor of π and denote the set of these by $\mathrm{ISuc}(\pi)$. Also for $\pi=ij\in M^{\rm r}$ and $d\in X^{\rm r}$, we say that d is *open* for π if $i<d<j$ and there is no $\pi'=pr\prec\pi$ with $p<d<r$, and denote the set of these by $\mathrm{Open}(\pi)$. And similarly for couples in $M^{\rm c}$ and elements of $X^{\rm c}$.

A current graph and its ingredients are identified with their images in the plane, and any edge in it is represented by a (directed) straight-line segment. We write (x_v,y_v) for the coordinates of a vertex (point) v, and say that an edge $e=(u,v)$ *points down* if $y_u>y_v$.

The initial graph H has the following features (seen from the construction in [5]).

(i) The "sources" $1,\ldots,k$ ("sinks" $1',\ldots,k'$) are disposed in this order from left to right in the upper (resp. lower) half of a circumference O, and the graph H is drawn within the circle (disc) O^* surrounded by O. (Strictly speaking, the construction of H that we describe is a mirror reflection of that in [5], which is more convenient for us and does not affect the construction in essence.)

(ii) Each couple $\pi=ij\in M^{\rm r}\cup M^{\rm c}$ is extended to a chord between the points i and j, which is subdivided into a path L_π whose edges are alternately forward and backward. Let R_π denote the region in O^* bounded by L_π and the paths $L_{\pi'}$ for $\pi'\in\mathrm{ISuc}(\pi)$. Then each edge e of H (regarded as a segment) having a point in the interior of R_π connects a vertex in L_π with either a vertex in $L_{\pi'}$ for some $\pi'\in\mathrm{ISuc}(\pi)$ or some vertex $d\in\mathrm{Open}(\pi)$. Moreover, e is directed to L_π if $\pi\in M^{\rm r}$, and from L_π if $\pi\in M^{\rm c}$.

(iii) Let R^* be the region in O^* bounded by the paths L_π for all maximal R- and C-couples π. Then any edge e of H having a point in the interior of R^* points down. Also if such an e has an incident vertex v lying on L_π for a maximal R-couple (resp. C-couple) π, then e leaves (resp. enters) v.

Using these properties, we transform H, step by step, keeping notation $H=(Z,U)$ for a current graph, and O^* for a current region (which is a deformed circle) containing H. Iteratively applied steps (S1) and (S2) are intended to make a graph with all edges pointing down.

(S1) Choose $\pi=ij\in M^{\rm r}$ and let $\overline{R}_\pi$ be the "upper half" of O^* bounded by L_π (then $\overline{R}_\pi$ contains L_π, the paths $L_{\pi'}$ for all $\pi'\prec\pi$, and the elements $d\in X^{\rm r}$ with $i<d<j$). We shift $\overline{R}_\pi-L_\pi$ upward by a sufficiently large distance $\lambda>0$. More precisely, each vertex $v\in Z$ lying in $\overline{R}_\pi-L_\pi$ is replaced by vertex v' with $x_{v'}=x_v$ and $y_{v'}=y_v+\lambda$. Each edge $(u,w)\in U$ of the old graph induces the corresponding edge of the new one, namely: edge (u',w') if both u,w lie in $\overline{R}_\pi-L_\pi$; edge (u,w) if $u,w\notin\overline{R}_\pi-L_\pi$; and edge (u', w) if $u \in \overline{R}_\pi - L_\pi$ and $w \in L_\pi$. (Case $u \in O^* - \overline{R}_\pi$ and $w \in \overline{R}_\pi$ is impossible.) Accordingly, the region O^* is enlarged by shifting the part $\overline{R}_\pi$ by $(0, \lambda)$ and filling the gap between L_π and $L_\pi + (0, \lambda)$ by the corresponding parallelogram.

One can realize that upon application of (S1) to all R-couples, the following property is ensured: for each $\pi \in M^{\rm r}$, all edges incident to exactly one vertex on L_π turn into edges pointing down. Moreover, since L_π is alternating and there is enough space (from below and from above) in a neighborhood of the current L_π, we can deform it into a zigzag path with all edges pointing down (by shifting each inner vertex v of L_π by a vector $(0, \epsilon)$ with an appropriate (positive or negative) $\epsilon \in \mathbb{R}$).

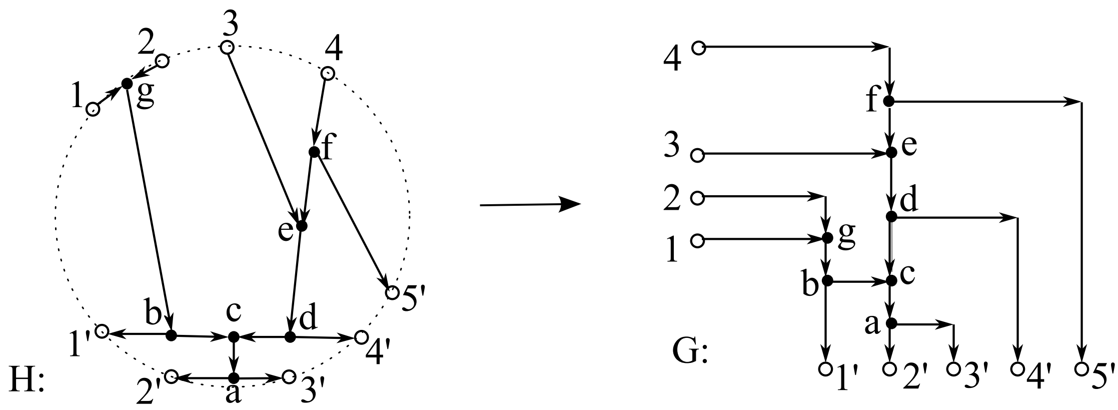
(S2) We choose $\pi \in M^{\rm c}$ and act similarly to (S1) with the differences that now $\overline{R}_\pi$ denotes the "lower half" of O^* bounded by L_π and that $\overline{R}_\pi$ is shifted downward (by a sufficiently large $\lambda > 0$).

Upon termination of the process for all R- and C-couples, all edges of the current graph H (which is homeomorphic to the initial one) point down, as required. Moreover, H has one more useful property: the sources $1, \ldots, k$ are "seen from above" and the sinks $1', \ldots, k'$ are "seen from below". Hence we can add to H "long" vertical edges $h_1, \ldots, h_k$ entering the vertices $1, \ldots, k$, respectively, and "long" vertical edges $h_{1'}, \ldots, h_{k'}$ leaving the vertices $1', \ldots, k'$, respectively, maintaining the planarity of the graph. In the new graph one should transfer the sources i into the beginnings of edges h_i, and the sinks i' into the ends of edges $h_{i'}$. One may assume that the new sources (sinks) lie within one horizontal line L (resp. L'), and that the rest of the graph lies below L and above L'.

Now we get rid of the edges (u, v) such that $x_u > x_v$ (i.e. "pointing to the left"), by making the linear transformation $v \mapsto v'$ for the points v in H, defined by $x_{v'} = x_v - \lambda y_v$ and $y_{v'} = y_v$ with a sufficiently large $\lambda > 0$.

Thus, we eventually obtain a graph H (homeomorphic to the initial one) in which there is no edges pointing up or to the left, and the sources and sinks are properly ordered from left to right in the lines L and L', respectively. Now it is routine to turn H into a grid-shaped graph G as required in the proposition. ∎

The transformation of H into G as in the proof is illustrated in the picture where $X^{\rm r} = \{4\}$, $Y^{\rm r} = \{1, 2, 3\}$, $X^{\rm c} = \emptyset$, $Y^{\rm c} = \{1', \ldots, 5'\}$, and $M = \{12, 1'4', 2'3', 35'\}$.



8 Concluding remarks

It looks reasonable to ask: how narrow is the class of UQ identities for minors of q-matrices compared with the class of those in the commutative version. We know that the latter class is formed by balanced families $\mathcal{I},\mathcal{K}$, whereas the former one is characterized via a stronger property of q-balancedness. So we can address the problem of characterizing the set of (homogeneous) balanced families $\mathcal{I},\mathcal{K} \subset \mathcal{E}^{m,n} \times \mathcal{E}^{m,n}$ that admit functions $\alpha : \mathcal{I} \to \mathbb{Z}$ and $\beta : \mathcal{K} \to \mathbb{Z}$ such that the quadruple $\mathcal{I},\mathcal{K},\alpha,\beta$ is q-balanced.

In an algorithmic setting, we address problem $(*)$: given $\mathcal{I},\mathcal{K}$ (as above), decide whether or not there exist corresponding α,β (as above). Concerning algorithmic complexity aspects, note that the number $|\mathbf{C}(\mathcal{I})|+|\mathbf{C}(\mathcal{K})|$ of configurations for $\mathcal{I},\mathcal{K}$ may be exponentially large compared with the number $|\mathcal{I}|+|\mathcal{K}|$ of corteges (since a cortege of size N may have $2^{O(N)}$ feasible matchings). Also both conditions of balancedness and q-balancedness involve bijections between $\mathbf{C}(\mathcal{I})$ and $\mathbf{C}(\mathcal{K})$. In light of this, it is logically reasonable to regard as the input of problem $(*)$ just the set $\mathbf{C}(\mathcal{I}) \sqcup \mathbf{C}(\mathcal{K})$ rather than $\mathcal{I} \sqcup \mathcal{K}$ (and measure the input size of $(*)$ accordingly). We conjecture that problem $(*)$ (specified in this way) is NP-hard and, moreover, it remains NP-hard even in the flag case.

The simplest example of balanced $\mathcal{I},\mathcal{K}$ for which problem $(*)$ has answer "not" arises in the flag case with $\mathcal{I},\mathcal{K}$ of cardinality one each. That is, we deal with quantized flag minors $[I]=[A|I]$ and $[K]=[B|K]$, where $A := \{1,\dots,|I|\}$ and $B := \{1,\dots,|J|\}$, and the (trivially balanced) families $\mathcal{I}$ and $\mathcal{K}$ consist of the single corteges $S=(A|I,B|J)$ and $T=(B|J,A|I)$, respectively. By Leclerc–Zelevinsky's result (Theorem 6.1), minors $[I]$ and $[J]$ quasicommute, i.e., satisfy (6.1), if and only if the sets I,J are weakly separated. We have explained how to obtain "if" part of this theorem by use of the flow-matching method, and now we explain how to use this method to show, relatively easily, "only if" part (which has a rather sophisticated proof in [12]).

So, assuming that I,J are not weakly separated, our aim is to show that there do not exist $\alpha(S),\beta(T) \in \mathbb{Z}$ satisfying the equality

$$\beta(T)-\alpha(S)=\zeta^{\circ}(S;M)-\zeta^{\bullet}(S;M) \tag{8.1}$$

for any feasible matching M for S. The crucial observation is that

(8.2) sets $I,J \subset [n]$ are not weakly separated if and only if S has two or more feasible matchings

(where "if" part, mentioned in 6.2, is trivial). In fact, we need a sharper version of (8.2): when $I,J \subset [n]$ are not weakly separated, there exist $M,M' \in \mathcal{M}(S)$ such that

$$\zeta^{\circ}(S;M)-\zeta^{\bullet}(S;M) \neq \zeta^{\circ}(S;M')-\zeta^{\bullet}(S;M'). \tag{8.3}$$

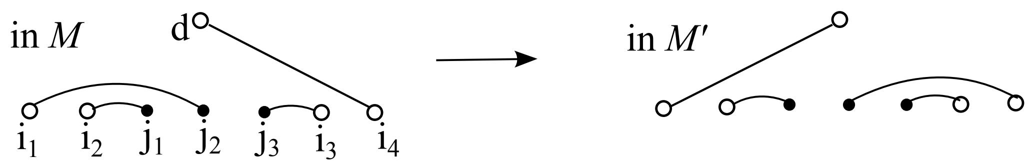
Then the fact that the exchange operation applied to S using either the whole M or the whole M' results in the same T implies that (8.1) cannot hold simultaneously for both M and M'.

To construct the desired M and M', we argue as follows. Let for definiteness $|I| \ge |J|$. From the fact that I, J are not weakly separated it is not difficult to conclude that there are $a, b \in [n]$ with $a < b$ such that the sets $\widetilde{I}^\circ := \{a, \dots, b\} \cap (I - J)$ and $\widetilde{J}^\bullet := \{a, \dots, b\} \cap (J - I)$ satisfy $|\widetilde{I}^\circ| - 1 = |\widetilde{J}^\bullet| =: k$, and $\widetilde{I}^\circ$ has a partition into nonempty sets I_1, I_2 satisfying $I_1 < \widetilde{J}^\bullet < I_2$. Let

$$I_1 = (i_1 < i_2 < \dots < i_p), \quad I_2 = (i_{p+1} < \dots < i_{k+1}), \quad \widetilde{J}^\bullet = (j_1 < \dots < j_k)$$

(then $i_p < j_1$ and $j_k < i_{p+1}$). Choose an arbitrary matching $M \in \mathcal{M}(S)$, and consider the set Π of couples in M containing elements of $\widetilde{J}^\bullet$; let $\Pi = \{\pi_\ell = \{j_\ell, i'_\ell\} \colon \ell = 1, \dots, k\}$. Each π_ℓ is a C-couple (since it cannot be an RC- couple, in view of $B - A = \emptyset$), and the feasibility condition (4.3) for M implies that only two cases are possible: (a) p couples in Π meet I_1 and the remaining $k - p$ couples meet I_2, and (b) $p - 1$ couples in Π meet I_1 and the remaining $k - p + 1$ couples meet I_2.

Suppose that case (a) takes place (then $\pi_\ell = \{j_\ell, i_{p-\ell+1}\}$ for $\ell = 1, \dots, p$, and $\pi_\ell = \{j_\ell, i_\ell\}$ for $\ell = p+1, \dots, k$). An especial role is played by the couple in M containing the last element i_{k+1} of I_2, say, $\pi = \{i_{k+1}, d\}$ (note that d belongs to either $A - B$ or $(J - I) - \widetilde{J}^\bullet$). We modify M by replacing the couple π by $\pi' := \{i_1, d\}$, and replacing $\pi_p = \{j_p, i_1\}$ by $\pi'_p := \{j_p, i_{k+1}\}$, forming matching M'. The picture illustrates the case $k = 3$, $p = 2$ and $d \in A - B$.



One can see that M' is feasible for S. Moreover, M and M' satisfy (8.3). Indeed, π_p contributes a unit to $\zeta^\circ(S; M)$ while π'_p contributes a unit to $\zeta^\bullet(S; M')$, the contributions from π and from π' are the same, and the rests of M and M' coincide.

Thus, in case (a), the single element families $\{S\}$ and $\{T\}$ along with any numbers $\alpha(S), \beta(T)$ are not q-balanced. Then the quasicommutation relation (6.1) (with any c) is impossible by Theorem 7.1. In case (b), the argument is similar. This yields the necessity ("only if" part) in Theorem 6.1.

In conclusion, it is tempting to ask: can one characterize the set of quasicommuting quantum minors in a general (non-flag) case? It turns out that such a characterization can be obtained, without big efforts, by use of the flow-matching method, yielding a generalization of Leclerc–Zelevinsky's result (Theorem 6.1).

Theorem 8.1 *Let $(I|J), (I'|J) \in \mathcal{E}^{m,n} \times \mathcal{E}^{m,n}$ and let $|I| \ge |I'|$. The following statements are equivalent:*

(i) *the minors $[I|J]$ and $[I'|J']$ quasicommute, i.e., $[I|J][I'|J'] = q^c[I'|J'][I|J]$ for some $c \in \mathbb{Z}$;*

(ii) *the cortege $S = (I|J, I'|J')$ admits exactly one feasible matching;*

(iii) *the sets* I, I' *are weakly separated, the sets* J, J' *are weakly separated, and for the refinement* $(I^\circ, I^\bullet, J^\circ, J^\bullet)$ *of* S*, one of the following takes place:*

(a) $|I^\bullet||J^\bullet| = 0$*; or*

(b) *both sets* $I^\bullet, J^\bullet$ *are nonempty, and either* $I^\circ < I^\bullet$ *and* $J^\bullet < J^\circ$*, or* $I^\bullet < I^\circ$ *and* $J^\circ < J^\bullet$*.*

Also in case (iii) the number c *is computed as follows: if* $I^\bullet = \emptyset$*,* $J^\bullet = J_1 \cup J_2$ *and* $J_1 < J^\circ < J_2$*, then* $c = |J_2| - |J_1|$*; (symmetrically) if* $J^\bullet = \emptyset$*,* $I^\bullet = I_1 \cup I_2$ *and* $I_1 < I^\circ < I_2$*, then* $c = |I_2| - |I_1|$*; if* $I^\circ < I^\bullet$ *and* $J^\bullet < J^\circ$*, then* $c = |I^\bullet| - |J^\bullet|$*; and (symmetrically) if* $I^\bullet < I^\circ$ *and* $J^\circ < J^\bullet$*, then* $c = |J^\bullet| - |I^\bullet|$*.*

Proof Implication (ii)→(i) is proved as in Section 6.2, and (iii)→(ii) is easy.

To show (i)→(iii), we use the fact that $|I^\circ| - |I^\bullet| = |J^\circ| - |J^\bullet| \geq 0$ (implied by $|I| \geq |I'|$), and note that a feasible matchings for S can be constructed by the following procedure (P) consisting of three steps. First, choose an arbitrary maximal feasible set $M^{\rm r}$ of R-couples in $Y^{\rm r} := I^\circ \cup I^\bullet$ (where the feasibility means that the elements of each couple have different colors and there is neither couples $\{i, j\}$ and $\{p, r\}$ with $i < p < j < r$, nor a couple $\{i, j\}$ and an element $d \in Y^{\rm r} - \cup(\pi \in M^{\rm r})$ with $i < d < j$; cf. (4.3)). Second, choose an arbitrary maximal feasible set $M^{\rm c}$ of C-couples in $Y^{\rm c} := J^\circ \cup J^\bullet$. Third, the remaining elements of $Y^{\rm r} \sqcup Y^{\rm c}$, which are all white (if exist), are coupled by a (unique) set $M^{\rm rc}$ of RC-couples. Then $M := M^{\rm r} \cup M^{\rm c} \cup M^{\rm rc}$ is a feasible matching for S.

Suppose that (iii) is false and consider possible cases.

1) Let J, J' be not weakly separated. Then we construct $M^{\rm r}, M^{\rm c}, M^{\rm rc}$ by procedure (P) and work with the matching $\widetilde{M} := M^{\rm c} \cup M^{\rm rc}$ in a similar way as in the above proof for the flag case (with non-weakly-separated column sets). This transforms $\widetilde{M}$ into $\widetilde{M}'$, and we obtain two different feasible matchings $M := \widetilde{M} \cup M^{\rm r}$ and $M' := \widetilde{M}' \cup M^{\rm r}$ for S satisfying (8.3). This yields violation of (i) (as well as (ii)) in the theorem. When I, I' are not weakly separated, the argument is similar.

2) Assume that both $I^\bullet, J^\bullet$ are nonempty. Then I°, J° are nonempty as well, and for the matching M formed by procedure (P), $M^{\rm r}$ covers $I^\bullet$, and $M^{\rm c}$ covers $J^\bullet$.

Denote by a, a' (resp. b, b') the minimal and maximal elements in $Y^{\rm r}$ (resp. $Y^{\rm c}$), respectively. Suppose that both a, b are black. Then we transforms M into M' by replacing the R-couple containing a, say, ad, and the C-couple containing b, say, bf, by the two RC-couples ab and df. It is easy to see that M' is feasible and M, M' satisfy (8.3) (as under the transformation $M \to M'$ the value $\zeta^\circ - \zeta^\bullet$ decreases by two), whence (i) is false. When both a', b' are black, we act similarly. So we may assume that each pair $\{a, b\}$ and $\{a', b'\}$ contains a white element. The case $a \in I^\circ$ and $b \in J^\circ$ is possible only if $|I^\circ| = |I^\bullet|$ (taking into account that $|I^\circ| \geq |Ib|$ and that $I^\circ, I^\bullet$, as well as $J^\circ, J^\bullet$, are weakly separated). But then $M^{\rm r}$ covers I° and J°, and we can construct a feasible matching $M' \neq M$ as in the previous case (by swapping the colors). And similarly when both a', b' are white.

Thus, we may assume that a, b have different colors, and so are a', b'. Suppose that $a, a' \in I^\circ$ and $b, b' \in J^\bullet$ (the case $a, a' \in I^\bullet$ and $b, b' \in J^\circ$ is similar). This is possible only if $|I^\circ| = |I^\bullet|$ (since $|I| \geq |J|$, and I, J are weakly separated). Then the feasible matching M constructed by (P) consists of only R- and C-couples. Take the R-couple in M containing a and the C-couple containing b', say, $\pi = \{a, i\}$ and $\pi' = \{j, b'\}$; then both a, j are white and both i, b' are black. Replace π, π' by the RC-couples $\{a, j\}$ and $\{i, b'\}$. It is easy to see that we obtain a feasible matching $M' \neq M$ satisfying (8.3).

The remaining situation is just as in (a) or (b) of (iii), and we can conclude with implication (i)→(iii). This completes the proof of the theorem. ▮

Remark 7. Note that the situation when $(I^\circ, I^\bullet, J^\circ, J^\bullet)$ has only one feasible matching can also be interpreted as follows. Let us change the colors of all elements in the upper half of the circumference O (i.e., I° becomes black and $I^\bullet$ becomes white). Then the quantities of white and black elements in O are equal and the elements of each color go in succession (cyclically).

Remark 8. When minors $[I|J]$ and $[I'|J']$ quasicommute with $c = 0$, we obtain the situation of "purely commuting" quantum minors, such as those discussed in Section 6.1. The last assertion in Theorem 8.1 enables us to completely characterize the set of corteges $(I|J, I'|J')$ determining commuting minors. Namely, only two cases are possible, in terms of the refinement $(I^\circ, I^\bullet, J^\circ, J^\bullet)$:

(C1) $|I^\circ| = |J^\circ|$ (as well as $I^\bullet| = |J^\bullet|$) and either $I^\circ < I^\bullet$ and $J^\bullet < J^\circ$, or, symmetrically, $I^\bullet < I^\circ$ and $J^\circ < J^\bullet$;

(C2) assuming for definiteness that $|I| \geq |I'|$, either $I^\bullet = \emptyset$ and $J^\bullet$ has a partition $J_1 \cup J_2$ such that $|J_1| = |J_2|$ and $J_1 < J^\circ < J_2$, or, symmetrically, $J^\bullet = \emptyset$ and $I^\bullet$ has a partition $I_1 \cup I_2$ such that $|I_1| = |I_2|$ and $I_1 < I^\circ < I_2$.

These cases are illustrated in the picture by two level diagrams.

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