On the combinatorial structure of crystals of types $\mathbf{A}, \mathbf{B}, \mathbf{C}$[1]

VLADIMIR I. DANILOV[2], ALEXANDER V. KARZANOV[3],
AND GLEB A. KOSHEVOY[2]

Abstract. Regular A_n-, B_n- and C_n-*crystals* are edge-colored directed graphs, with ordered colors $1, 2, \ldots, n$, which are related to representations of quantized algebras $U_q(\mathfrak{sl}_{n+1})$, $U_q(\mathfrak{sp}_{2n})$ and $U_q(\mathfrak{so}_{2n+1})$, respectively. We develop combinatorial methods to reveal refined structural properties of such objects.

Firstly, we study subcrystals of a regular A_n-crystal K and characterize pairwise intersections of maximal subcrystals with colors $1, \ldots, n-1$ and colors $2, \ldots, n$. This leads to a recursive description of the structure of K and provides an efficient procedure of assembling K.

Secondly, using merely combinatorial means, we demonstrate a relationship between regular B_n-crystals (resp. C_n-crystals) and regular *symmetric* A_{2n-1}-crystals (resp. A_{2n}-crystals).

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1   Introduction

Crystals are certain "exotic" edge-colored graphs. This graph-theoretic abstraction, introduced by Kashiwara [6, 7], has proved its usefulness in the theory of representations of Lie algebras and their quantum analogues. A (general) *crystal* is a directed graph K such that: the edges are partitioned into n subsets, or *color* classes, labeled $1, \ldots, n$, each connected monochromatic subgraph of K is a finite path, and there is an interrelation between the lengths of such paths described in terms of the $n \times n$ Cartan matrix $M = (m_{ij})$ related to a given Lie algebra $\mathfrak{g}$. This interrelation is: for colors i, j, any edge (u, v) with color i satisfies $(h_j(u) - t_j(u)) - (h_j(v) - t_j(v)) = m_{ij}$, where for a vertex v', $h_j(v')$ (resp. $t_j(v')$) denotes the length of the maximal path colored j that begins (resp. ends) at v'. Throughout we assume, w.l.o.g., that any crystal in question is (weakly) connected, and call an edge with color i an *i-edge*. Depending on Cartan matrices, several *types* of crystals are distinguished.

Of most interest are crystals of representations, or *regular* crystals. They are associated to elements of a certain basis of the highest weight integrable modules (representations) over a quantized algebra $U_q(\mathfrak{g})$. There are known "global" models to characterize the regular crystals for a variety of types: generalized Young tableaux [8], Lusztig's canonical bases [11], Littelmann's path model [9, 10], and some others.

This paper continues our combinatorial study of crystals begun in [1, 2, 3] and considers n-colored regular crystals of three types: A, B, C, where the number n of colors is arbitrary. Recall that *type A* (concerning $\mathfrak{g} = \mathfrak{sl}_{n+1}$) is related to the Cartan matric M with: $m_{ij} = -1$ if $|i-j| = 1$, $m_{ij} = 0$ if $|i-j| > 1$, and $m_{ii} = 2$. For *type B* (concerning $\mathfrak{g} = \mathfrak{sp}_{2n}$), the matrix is obtained from the above M by replacing $m_{n-1,n}$ by -2. And for *type C* (concerning $\mathfrak{g} = \mathfrak{so}_{2n+1}$), one should replace $m_{n,n-1}$ by -2. We will refer to a regular n-colored crystal of type A (B, C) as an A_n-*crystal* (resp. B_n-, C_n-*crystal*) and omit the index n when the number of colors is not specified.

It is known that the (finite) regular crystals K of these types have the following properties. (i) K is acyclic (i.e. without directed cycles) and has exactly one zero-indegree vertex, called the *source*, and exactly one zero-outdegree vertex, called the *sink* of K. (ii) For any $I \subseteq \{1, \ldots, n\}$, each (inclusion-wise) maximal connected subgraph of K whose edges have colors from I is a regular crystal related to the corresponding $I \times I$ submatrix of the Cartan matrix of K. Throughout, speaking of a subcrystal of K, we will always mean a subgraph of this kind.

Two-colored subcrystals are of most importance, due to the result in [5] that for a crystal with exactly one zero-indegree vertex, the regularity of all such subcrystals implies the regularity of the whole crystal. Let K' be a two-colored subcrystal with colors i, j in K. Then for type A, K' is the Cartesian product of a path with color i and a path with color j (forming an $A_1 \times A_1$-crystal) when $|i-j| > 1$, and an A_2-crystal when $|i-j| = 1$. For type B, the only difference is that K' is a B_2-crystal when $(i, j) = (n-1, n)$, and the corresponding submatrix is viewed as $\left(\begin{smallmatrix} 2 & -2 \\ -1 & 2 \end{smallmatrix}\right)$. And for type C, K' with $(i, j) = (n-1, n)$ is again a B_2-crystal but the corresponding submatrix is now $\left(\begin{smallmatrix} 2 & -1 \\ -2 & 2 \end{smallmatrix}\right)$. The A-crystals belong to the group of *simply-laced* crystals (defined by the requirement that each two-colored subcrystal is of type $A_1 \times A_1$ or A_2), and the B- and C-crystals belong to the group of *doubly-laced* ones (where each two-colored subcrystal is of type $A_1 \times A_1$ or A_2 or B_2); cf., e.g., [13].

Throughout the paper we are going to deal with regular crystals only, and for this reason the adjective "regular" will usually be omitted. It should be noted that even in case of A_2- and B_2-crystals, the corresponding specifications of "global" models from [8, 9, 10, 11] are rather intricate to work with directly. Fortunately, in the last decade there appeared more explicit and enlightening ways to define these crystals, via "local" graph-theoretic axioms or by use of direct combinatorial constructions. In case of A_2-crystals, a short list of "local" defining axioms is pointed out by Stembridge [13] and an explicit construction is given in [1]. According to that construction, any A_2-crystal can be obtained from an $A_1 \times A_1$-crystal by replacing each monochromatic path of the latter by a graph viewed as a triangular half of a directed square grid. In case of B_2-crystals, both "local" axioms and a direct combinatorial construction are

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given in [3]. It is shown there that a B_2-crystal can be obtained from an A_2-crystal by replacing each monochromatic path by a certain quadrangular part of a square grid. Also [3] describes an alternative combinatorial construction for B_2-crystals, the so-called *worm model.* This model will be extensively used in this paper. (For some other results on B_2-crystals, see [14].)

An important fact is that for any n-tuple $c = (c_1, \ldots, c_n)$ of nonnegative integers, there exists exactly one A_n-crystal K such that each c_i is equal to the length of the maximal path with color i beginning at the source (for a short combinatorial proof, see [2, Sec. 2]). A similar property takes place for B_n- and C_n-crystals. We denote a crystal K (of a given type) determined by c in this way by $K(c)$, and refer to c as the *parameter* of this crystal.

When $n > 2$, the combinatorial structure of A_n-crystals becomes rather compli-cated, even for $n = 3$. Attempting to learn more about this structure, we elaborated in [2] a new combinatorial construction, the so-called *crossing model* (which is a re-finement of the Gelfand-Tsetlin pattern model [4]). This powerful tool has helped us to reveal more structural features of an A_n-crystal $K = K(c)$. In particular, K has the so-called *principal lattice*, a set Π of vertices with the following nice properties:

(P1) Π contains the source and sink of K, and the vertices $v \in \Pi$ correspond to the elements of the integer box $\mathcal{B}(c) := \{a \in \mathbb{Z}^n \colon 0 \le a \le c\}$; we write $v = \check{v}[a]$;

(P2) For any $a, a' \in \mathcal{B}(c)$ with $a \le a'$, the *interval* of K from $\check{v}[a]$ to $\check{v}[a']$ (i.e. the subgraph of K formed by the vertices and edges contained in (directed) paths from $\check{v}[a]$ to $\check{v}[a']$) is isomorphic to the A_n-crystal $K(a' - a)$, and its principal lattice consists of the principal vertices $\check{v}[a'']$ of K with $a \le a'' \le a'$;

(P3) The set $\mathcal{K}^{(-n)}$ of $(n-1)$-colored subcrystals K' of K having colors $1, \ldots, n-1$ is bijective to Π; more precisely, $K' \cap \Pi$ consists of a single vertex (called the *heart* of K' w.r.t. K); and similarly for the set $\mathcal{K}^{(-1)}$ of subcrystals of K with colors $2, \ldots, n$.

(A sort of principal lattice can be introduced for B- and C-crystals as well; it satisfies (P1) and (P2) but not (P3); see Remark 5 in the end of Section 8.)

For $a \in \mathcal{B}(c)$, let $K^{\uparrow}[a]$ (resp. $K^{\downarrow}[a]$) denote the subcrystal in $\mathcal{K}^{(-n)}$ (resp. in $\mathcal{K}^{(-1)}$) that contains the principal vertex $\check{v}[a]$; we call it the *upper* (resp. *lower*) subcrystal at a. It is shown in [2] that the parameter of this subcrystal is expressed by a linear function of c and a, and that the number of upper (lower) subcrystals with a fixed parameter c' is expressed by a piece-wise linear function of c and c'.

In this paper, we further use the crossing model, aiming to obtain a refined descrip-tion of the structure of an A_n-crystal K. We study the intersections of subcrystals $K^{\uparrow}[a]$ and $K^{\downarrow}[b]$ for all $a, b \in \mathcal{B}(c)$. This intersection may be empty or consist of one or more subcrystals with colors $2, \ldots, n-1$, called *middle* subcrystals of K. Each of these middle subcrystals $\widetilde{K}$ is therefore a lower subcrystal of $K^{\uparrow}[a]$ and an upper subcrystal of $K^{\downarrow}[b]$; so $\widetilde{K}$ has a unique vertex z in the principal lattice $\Pi^{\uparrow}$ of the for-mer, and a unique vertex z' in the principal lattice $\Pi^{\downarrow}$ of the latter. Our main result on A-crystals (Theorem 3.1) and its consequences give explicit relations between a, b, the locus of z in $\Pi^{\uparrow}$, the locus of z' in $\Pi^{\downarrow}$, and the parameters of K and $\widetilde{K}$.

This gives rise to a recursive procedure of assembling of the A_n-crystal $K(c)$. More precisely, suppose that the $(n-1)$-colored crystals $K^\uparrow[a]$ and $K^\downarrow[b]$ for all $a, b \in \mathcal{B}(c)$ are already constructed. Then we can combine these subcrystals to obtain the desired crystal $K(c)$, by properly identifying the corresponding middle subcrystals (if any) for each pair $K^\uparrow[a], K^\downarrow[b]$. This recursive method is implemented as an efficient algorithm which, given a parameter $c \in \mathbb{Z}_+^n$, outputs the crystal $K(c)$. The running time of the algorithm and the needed space are bounded by $Cn^2|K(c)|$, where C is a constant and $|K(c)|$ is the size of $K(c)$. (It may be of practical use for small n and c; in general, an A_n-crystal has "dimension" $\frac{n(n+1)}{2}$ and its size grows sharply by increasing c.)

The second part of the paper is devoted to n-colored (regular) B-crystals. With the help of Theorem 3.1, we explain, using merely combinatorial means, that any B-crystal can be extracted from a symmetric A-crystal. More precisely, given $c \in \mathbb{Z}_+^n$, define the $(2n-1)$-tuple c' by $c_i' = c_{2n-i}' := c_i$ for $i = 1, \ldots, n$. The A_{2n-1}-crystal $K = K(c')$ has a canonical involution σ on the vertices under which the image $(\sigma(u), \sigma(v))$ of an i-edge (u, v) is a $(2n-i)$-edge. We say that K is *symmetric* and that a vertex v with $\sigma(v) = v$ is *self-complementary*; let S be the set of such vertices. The *symmetric extract* from K is the n-colored graph $\widetilde{K}$ whose vertex set is S and whose edges are defined as follows: (i) the edges of $\widetilde{K}$ colored n are exactly the n-edges of K connecting elements of S, and (ii) for $i < n$, vertices $u, v \in S$ are connected in $\widetilde{K}$ by edge (u, v) colored i if K contains a 2-edge path from u to v whose edges are colored i and $(2n-i)$. We prove that $\widetilde{K}$ is isomorphic to the B_n-crystal with the parameter c. The crucial part of the proof is a verification in case $n = 2$.

In the final part, we explain a similar fact for C-crystals; now the C_n-crystals are extracted from symmetric A_{2n}-crystals.

It should be noted that such a way of constructing B- and C-crystals from corresponding symmetric A-crystals has been known; this can be concluded from the work of Naito and Sagaki [12] where the argument relies on a sophisticated path model. Our goal is to give alternative proofs which are direct and purely combinatorial. We take advantages from rather transparent axiomatics and constructions for crystals of types A,B,C, and appeal to structural results from Section 3.

This paper is organized as follows. Section 2 is devoted to basic definitions and backgrounds. Here we recall "local" axioms and the crossing model for A-crystals, and review needed results on the principal lattice Π of an A_n-crystal and relations between Π and the $(n-1)$-colored subcrystals from [2]. Section 3 gives a recursive description of the structure of an A_n-crystal K and the algorithm of assembling K; here we rely on the main structural result (Theorem 3.1) proved in the next Section 4. The devised assembling method is illustrated in Section 5 for two special cases of A-crystals: for an arbitrary A_2-crystal (in which case the method can be compared with the explicit combinatorial construction in [1]), and for the particular A_3-crystal $K(1, 1, 1)$. The rest of the paper is devoted to B- and C-crystals. Our combinatorial proof of the theorem that the B_n-crystals are exactly the extracts from symmetric A_{2n-1}-crystals is given in Sections 6–8. Here Section 6 reduces the task to $n = 2$, Section 7 recalls

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the worm model from [3], and the crucial Section 8 gives a proof for $n = 2$, relying on the construction of B_2-crystals via the worm model. An important step in the proof consists in representing the self-complementary vertices of a symmetric A_3-crystal as integer points of a certain 4-dimensional polytope (in Theorem 8.2). Arguing in a similar fashion, Section 9 gives a combinatorial proof of the theorem that the extracts from symmetric A_{2n}-crystals are C_n-crystals. Most technical claims used in Sections 8 and 9 are proved in the Appendix.

2   Preliminaries

In this section we recall definitions and some basic properties of (regular) crystals of types A, B and C, referring to them as A-, B- and C-*crystals*, respectively, and review results from [2] that will be important for further purposes.

An *n-colored crystal* is a certain directed graph K whose edge set $E(K)$ is partitioned into n subsets $E_1, \ldots, E_n$, denoted as $K = (V(K), E_1 \sqcup \ldots \sqcup E_n)$. We assume that K is (weakly) *connected*, i.e. it is not the disjoint union of two nonempty graphs. We say that an edge $e \in E_i$ has *color i*, or is an *i-edge*. When speaking of a *subcrystal* K' of K, we always mean that K' is inclusion-wise *maximal* among the connected subgraphs having the same set of colors as K'.

2.1   Crystals of type A

Stembridge [13] pointed out a list of "local" graph-theoretic axioms for the regular simply-laced crystals. The A-crystals form a subclass of those and are defined by axioms (A1)–(A5) below; we give the axiomatics in a slightly different, but equivalent, form compared with [13]. Let K be an n-edge-colored graph as before.

Unless explicitly stated otherwise, by a *path* we mean a simple finite directed path, i.e. a sequence of the form $(v_0, e_1, v_1, \ldots, e_k, v_k)$, where $v_0, v_1, \ldots, v_k$ are distinct vertices and each e_i is an edge from v_{i-1} to v_i (admitting $k = 0$).

The first axiom concerns the structure of monochromatic subgraphs of K.

(A1) For $i = 1, \ldots, n$, each connected subgraph of $(V(K), E_i)$ is a path.

So each vertex of K has at most one incoming i-edge and at most one outgoing i-edge, and therefore one can associate to the set E_i a *partial invertible operator F_i* acting on vertices: (u, v) is an i-edge if and only if F_i *acts at u* and $F_i(u) = v$ (or $u = F_i^{-1}(v)$, where F_i^{-1} is the partial operator inverse to F_i). Since K is connected, one can use the operator notation to express any vertex via another one. For example, the expression $F_1^{-1} F_3^2 F_2(v)$ determines the vertex w obtained from a vertex v by traversing 2-edge (v, v'), followed by traversing 3-edges (v', u) and (u, u'), followed by traversing 1-edge (w, u') in backward direction. Emphasize that every time we use such an operator expression in what follows, this automatically says that all corresponding edges do exist in K.

We refer to a monochromatic path with color i on the edges as an i-*path*, and to a maximal i-path as an i-*line* (the latter is an A_1-subcrystal of K). The i-line passing through a given vertex v (possibly consisting of the only vertex v) is denoted by $P_i(v)$, its part from the first vertex to v by $P_i^{\mathrm{in}}(v)$, and its part from v to the last vertex by $P_i^{\mathrm{out}}(v)$ (the *tail* and *head* parts of P w.r.t. v). The lengths (i.e. the numbers of edges) of $P_i^{\mathrm{in}}(v)$ and $P_i^{\mathrm{out}}(v)$ are denoted by $t_i(v)$ and $h_i(v)$, respectively.

Axioms (A2)–(A5) concern interrelations of different colors i, j. They say that each component of the two-colored graph $(V(K), E_i \sqcup E_j)$ forms an A_2-crystal when colors i, j are *neighboring*, which means that $|i - j| = 1$, and forms an $A_1 \times A_1$-crystal otherwise.

When an edge of a color i is traversed, the head and tail part lengths of lines of another color j behave as follows:

(A2) For different colors i, j and for an edge (u, v) with color i, one holds $t_j(v) \leq t_j(u)$ and $h_j(v) \geq h_j(u)$. The value $(h_j(u) - t_j(u)) - (h_j(v) - t_j(v))$ is the constant m_{ij} equal to -1 if $|i - j| = 1$, and 0 otherwise. Furthermore, h_j is convex on each i-path, in the sense that if $(u, v), (v, w)$ are consecutive i-edges, then $h_j(u) + h_j(w) \geq 2h_j(v)$.

These constants m_{ij} are just the coefficients of the Cartan $n \times n$ matrix M related to the crystal type A and the number n of colors. Each diagonal entry m_{ii} equals 2, which agrees with the trivial relation $(h_i(u) - t_i(u)) - (h_i(v) - t_i(v)) = 2$ for an i-edge (u, v).

It follows that for neighboring colors i, j, each i-line P contains a unique vertex r such that: when traversing any edge e of P before r (i.e. $e \in P_i^{\mathrm{in}}(r)$), the tail length t_j decreases by 1 while the head length h_j does not change, and when traversing any edge of P after r, t_j does not change while h_j increases by 1. This r is called the *critical* vertex for P, i, j. To each i-edge $e = (u, v)$ we associate *label* $\ell_j(e) := h_j(v) - h_j(u)$; then $\ell_j(e) \in \{0, 1\}$ and $t_j(v) = t_j(u) - 1 + \ell_j(e)$. Emphasize that the critical vertices on an i-line P w.r.t. its neighboring colors $j = i - 1$ and $j = i + 1$ may be different (and so are the edge labels on P).

Two operators $F = F_i^\alpha$ and $F' = F_j^\beta$, where $\alpha, \beta \in \{1, -1\}$, are said to *commute* at a vertex v if each of F, F' acts at v (i.e. corresponding i-edge and j-edge incident with v exist) and $FF'(v) = F'F(v)$. The third axiom indicates situations when such operators commute for neighboring i, j.

(A3) Let $|i - j| = 1$. (a) If a vertex u has outgoing i-edge (u, v) and outgoing j-edge (u, v') and if $\ell_j(u, v) = 0$, then $\ell_i(u, v') = 1$ and F_i, F_j commute at v. Symmetrically: (b) if a vertex v has incoming i-edge (u, v) and incoming j-edge (u', v) and if $\ell_j(u, v) = 1$, then $\ell_i(u', v) = 0$ and F_i^{-1}, F_j^{-1} commute at v. (See the picture.)

Using this axiom, one easily shows that if four vertices are connected by two i-edges e, e' and two j-edges $\widetilde{e}, \widetilde{e}'$ (forming a "square"), then $\ell_j(e) = \ell_j(e') \neq \ell_i(\widetilde{e}) = \ell_i(\widetilde{e}')$ (as illustrated in the picture). Another important consequence of (A3) is that for neighboring colors i, j, if v is the critical vertex on an i-line w.r.t. color j, then v is also the critical vertex on the j-line passing v w.r.t. color i, i.e. we can speak of common critical vertices for the pair $\{i, j\}$.

The fourth axiom points out situations when, for neighboring i, j, the operators F_i, F_j and their inverse ones "remotely commute" (they are said to satisfy the "Verma relation of degree 4").

(A4) Let $|i - j| = 1$. (i) If a vertex u has outgoing edges with color i and color j and if each edge is labeled 1 w.r.t. the other color, then $F_i F_j^2 F_i(u) = F_j F_i^2 F_j(u)$. Symmetrically: (ii) if v has incoming edges with color i and color j and if both are labeled 0, then $F_i^{-1}(F_j^{-1})^2 F_i^{-1}(v) = F_j^{-1}(F_i^{-1})^2 F_j^{-1}(v)$. (See the picture.)
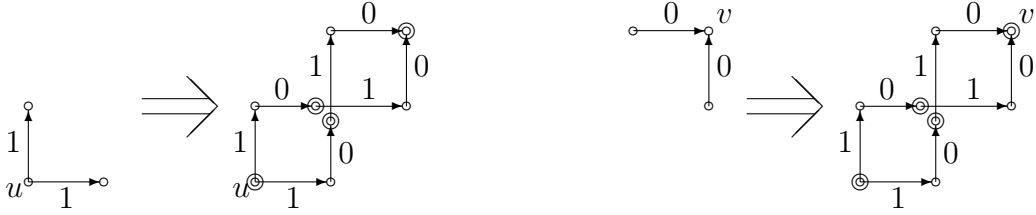
Again, one shows that the label w.r.t. i, j of each of the eight involved edges is determined uniquely, just as indicated in the above picture (where the bigger circles indicate critical vertices).

The final axiom concerns non-neighboring colors.

(A5) Let $|i - j| \geq 2$. Then for any $F \in \{F_i, F_i^{-1}\}$ and $F' \in \{F_j, F_j^{-1}\}$, the operators F, F' commute at each vertex where both act.

This is equivalent to saying that each component of the two-colored subgraph $(V(K), E_i \sqcup E_j)$ is the Cartesian product of an i-path P and a j-path P', or that each subcrystal of K with non-neighboring colors i, j is an $A_1 \times A_1$-crystal.

One shows that any A_n-crystal K is finite and has exactly one zero-indegree vertex s_K and one zero-outdegree vertex t_K, called the *source* and *sink* of K, respectively. Furthermore, the A_n-crystals K admit a nice parameterization: the lengths $h_1(s_K), \ldots, h_n(s_K)$ of monochromatic paths determine K, and for each tuple $c = (c_1, \ldots, c_n)$ of nonnegative integers, there exists a (unique) A_n-crystal K such that $c_i = h_i(s_K)$ for $i = 1, \ldots, n$. (See [13] and [2].) We call c the *parameter* of K and denote K by $K(c)$.

2.2 Crystals of types B and C

These crystals are defined via the types of their two-colored subcrystals, exhibited in axioms (BC1)–(BC3). The difference between B- and C-crystals concerns only specifications of axiom (BC3) given in (BC4) and (BC4′). As before, $K = (V(K), E_1 \sqcup \ldots \sqcup E_n)$ is a connected n-colored graph.

(BC1) K satisfies (A1) and (A5).

(BC2) For colors $i, j < n$ with $|i - j| = 1$, each component of $(V(K), E_i \sqcup E_j)$ is an A_2-crystal, i.e. it satisfies (A2)–(A4).

(BC3) Each component of $(V(K), E_{n-1} \sqcup E_n)$ is isomorphic to a B_2-crystal.

There are several ways to define B_2-crystals. Based on Littlemann's path model [10], it is shown in [3] that a B_2-crystal can be equivalently defined in three ways: (i) via an explicit combinatorial construction, (ii) via a graphical *worm model*, which represents each vertex of the crystal as a certain pair of line-segments in a rectangle, and (iii) via a list of 14 local or "almost local" axioms. Compared with the A_2 case, this list is big enough and less convenient to handle practically. In contrast, the worm model has a rather compact description, reviewed in Section 7, and we will appeal just to this model in our examination of two-colored symmetric extracts from corresponding A-crystals (in Sections 8 and 9).

For a B_2-crystal, with colors i and j say, the coefficients m_{ij} and m_{ji} are different and take values -1 and -2 (where, as before, $m_{pq} = (h_q(u) - t_q(u)) - (h_q(v) - t_q(v))$ for an edge (u, v) of color p). The difference between B and C types is the following:

(BC4) For B_n-crystals, $m_{n-1,n} = -2$ and $m_{n,n-1} = -1$.

(BC4′) For C_n-crystals, $m_{n-1,n} = -1$ and $m_{n,n-1} = -2$.

The Cartan matrices for types A, B, C and $n = 4$ are illustrated in the picture where the coefficient in each empty cell is zero.

A:

2	−1		
−1	2	−1	
	−1	2	−1
		−1	2

B:

2	−1		
−1	2	−1	
	−1	2	−2
		−1	2

C:

2	−1		
−1	2	−1	
	−1	2	−1
		−2	2

Using arguments as in [2, 13] for A-crystals, one can show that any B-crystal K is finite, has exactly one source $s = s_K$ (and one sink), and is determined by the lengths $h_1(s_K), \ldots, h_n(s_K)$. Also a B_n-crystal K with $h(s_K) = c$ exists for any $c \in \mathbb{Z}_+^n$, and similarly for C_n-crystals (this is explained in [3] for $n = 2$, and follows from reasonings in Sections 6 and 9 for $n > 2$). This gives a parametrization of B-crystals similar to that for A-crystals.

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2.3 The crossing model for A_n-crystals

Following [2], the *crossing model* $\mathcal{M}_n(c)$ generating the A_n-crystal $K = K(c)$ with a parameter $c = (c_1, \ldots, c_n) \in \mathbb{Z}_+^n$ consists of three ingredients:

(i) a directed graph $G_n = G = (V(G), E(G))$ depending on n, called the *supporting graph* of the model;

(ii) a set $\mathcal{F} = \mathcal{F}(c)$ of *feasible* functions on $V(G)$;

(iii) a set $\mathcal{E} = \mathcal{E}(c)$ of transformations $f \mapsto f'$ of feasible functions, called *moves* in the model.

To explain the construction of the supporting graph G, we first introduce another directed graph $\mathcal{G} = \mathcal{G}_n$ that we call the *proto-graph* of G. Its node set consists of elements $V_i(j)$ for all $i, j \in \{1, \ldots, n\}$ such that $j \leq i$. (We use the term "node" for vertices in the crossing model, to avoid a possible mess between these and vertices of crystals.) Its edges are all possible pairs of the form $(V_i(j), V_{i-1}(j))$ (*ascending* edges) or $(V_i(j), V_{i+1}(j+1))$ (*descending* edges). We say that the nodes $V_i(1), \ldots, V_i(i)$ form i-th *level* of $\mathcal{G}$ and order them as indicated (by increasing j). We visualize $\mathcal{G}$ by drawing it on the plane so that the nodes of the same level lie in a horizontal line, the ascending edges point North-East, and the descending edges point South-East. See the picture where $n = 4$.

$$V_1(1)$$
$$V_2(1) \qquad V_2(2)$$
$$V_3(1) \qquad V_3(2) \qquad V_3(3)$$
$$V_4(1) \qquad V_4(2) \qquad V_4(3) \qquad V_4(4)$$
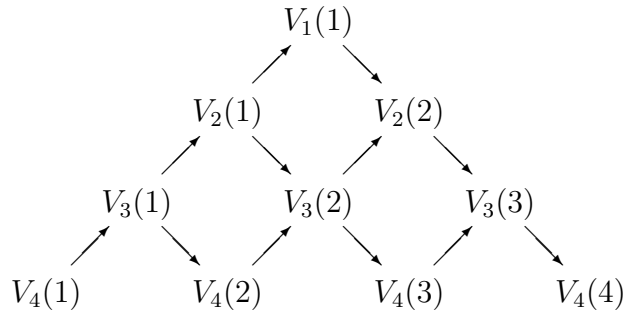
The supporting graph G is produced by replicating elements of $\mathcal{G}$ as follows. Each node $V_i(j)$ generates $n-i+1$ nodes of G, denoted as $v_i^k(j)$ for $k = i-j+1, \ldots n-j+1$, which are ordered by increasing k (and accordingly follow from left to right in the visualization). We identify $V_i(j)$ with the set of these nodes and call it a *multinode* of G. Each edge of $\mathcal{G}$ generates a set of edges of G (a *multi-edge*) connecting elements with equal upper indices. More precisely, $(V_i(j), V_{i-1}(j))$ produces $n-i+1$ ascending edges $(v_i^k(j), v_{i-1}^k(j))$ for $k = i-j+1, \ldots, n-j+1$, and $(V_i(j), V_{i+1}(j+1))$ produces $n-i$ descending edges $(v_i^k(j), v_{i+1}^k(j+1))$ for $k = i-j+1, \ldots, n-j$.

The resulting G is the disjoint union of n directed graphs $G^1, \ldots, G^n$, where each G^k contains all vertices of the form $v_i^k(j)$. Also G^k is isomorphic to the Cartesian product of two paths, with the lengths $k-1$ and $n-k$. For example, for $n = 4$, the graph G is viewed as
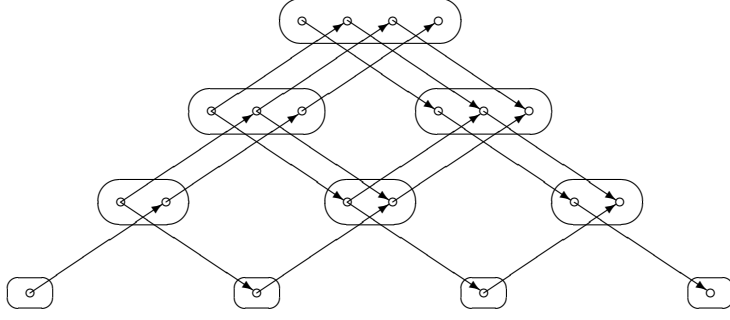
(where the multinodes are surrounded by ovals) and its components G^1, G^2, G^3, G^4 are viewed as
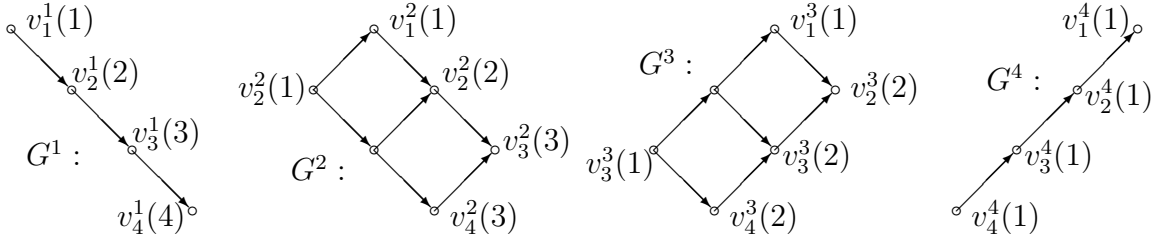
So each node $v = v_i^k(j)$ of G has at most four incident edges, namely, $(v_{i-1}^k(j-1), v)$, $(v_{i+1}^k(j), v)$, $(v, v_{i-1}^k(j))$, $(v, v_{i+1}^k(j+1))$; we refer to them, when exist, as the NW-, SW-, NE-, and SE-*edges*, and denote by $e^{\text{NW}}(v), e^{\text{SW}}(v), e^{\text{NE}}(v), e^{\text{SE}}(v)$, respectively.

By a *feasible* function in the model (with a given c) we mean a function $f : V(G) \to \mathbb{Z}_+$ satisfying the following three conditions, where for an edge $e = (u, v)$, $\partial f(e)$ denotes the increment $f(u) - f(v)$ of f on e, and e is called *tight* for f, or f-*tight*, if $\partial f(e) = 0$:

(2.1)   (i) f is *monotone* on the edges, in the sense that $\partial f(e) \geq 0$ for all $e \in E(G)$;

     (ii) $0 \leq f(v) \leq c_k$ for each $v \in V(G^k)$, $k = 1, \ldots, n$;

     (iii) each multinode $V_i(j)$ contains a node v with the following property: the edge $e^{\text{SE}}(u)$ is tight for each node $u \in V_i(j)$ preceding v, and $e^{\text{SW}}(u')$ is tight for each node $u' \in V_i(j)$ succeeding v.

The *first* node $v = v_i^k(j)$ (i.e. with k minimum) satisfying the property in (iii) is called the *switch-node* of the multinode $V_i(j)$. These nodes play an important role in our transformations of feasible functions in the model.

To describe the rule of transforming $f \in \mathcal{F}(c)$, we first extend each G^k by adding extra nodes and edges (following [2] and aiming to slightly simplify the description). In the extended directed graph $\bar{G}^k$, the node set consists of elements $v_i^k(j)$ for all $i = 0, \ldots, n+1$ and $j = 0, \ldots, n$ such that $j \leq i$. The edge set of $\bar{G}^k$ consists of all possible pairs of the form $(v_i^k(j), v_{i-1}^k(j))$ or $(v_i^k(j), v_{i+1}^k(j+1))$. Then all $\bar{G}^k$ are isomorphic. The disjoint union of these $\bar{G}^k$ gives the *extended supporting graph* $\bar{G}$.

Each feasible function on $V(G)$ is extended to the extra nodes $v = v_i^k(j)$ as follows: $f(v) := c_k$ if there is a path from v to a node of G^k, and $f(v) := 0$ otherwise (one

may say that v lies on the left of G^k in the former case, and on the right of G^k in the latter case). In particular, each edge e of $\bar{G}$ not incident with a node of G is tight, i.e. $\partial f(e) = 0$ (extending ∂f to the extra edges). For a node $v = v_i^k(j)$ with $1 \le j \le i \le n$, define the value $\varepsilon(v) = \varepsilon_f(v)$ by

$$\varepsilon(v) := \partial f(e^{\mathrm{NW}}(v)) - \partial f(e^{\mathrm{SE}}(u)) \quad (= \partial f(e^{\mathrm{SW}}(v)) - \partial f(e^{\mathrm{NE}}(u)), \qquad (2.2)$$

where $u := v_i^k(j-1)$. For a multinode $V_i(j)$ (and the given f), define the numbers

$$\varepsilon_i(j) := \sum (\varepsilon(v) \colon v \in V_i(j)) \qquad (2.3)$$

and

$$\widetilde{\varepsilon}_i(j) := \max\{0, \min\{\varepsilon_i(p) + \varepsilon_i(p+1) + \ldots + \varepsilon_i(j) \colon 1 \le p \le j\}\}. \qquad (2.4)$$

We call $\varepsilon(v)$, $\varepsilon_i(j)$ and $\widetilde{\varepsilon}_i(j)$ the *slack* at a node v, the *total slack* at a multinode $V_i(j)$ and the *reduced slack* at $V_i(j)$, respectively. (We define the slacks $\varepsilon, \widetilde{\varepsilon}$ in a slightly different way than in [2], which however does not affect the definitions of active multinodes and switch-nodes below.)

Now we are ready to define the transformations of f (or the moves from f). At most n transformations $\phi_1, \ldots, \phi_n$ are possible. Each ϕ_i changes f within level i and is applicable when this level contains a multinode $V_i(j')$ with $\widetilde{\varepsilon}_i(j') > 0$. In this case we take the multinode $V_i(j)$ such that

$$\widetilde{\varepsilon}_i(j) > 0 \quad \text{and} \quad \widetilde{\varepsilon}_i(q) = 0 \ \text{ for } q = j+1, \ldots, i, \qquad (2.5)$$

referring to it as the *active* multinode for the given f and i, and increase f by 1 at the *switch-node* in $V_i(j)$, preserving f on the other nodes of G. It is shown [2] that the resulting function $\phi_i(f)$ is again feasible.

So the model generates the n-colored directed graph $\mathcal{K}(c) = (\mathcal{F}, \mathcal{E}_1 \sqcup \ldots \sqcup \mathcal{E}_n)$, where each color class $\mathcal{E}_i$ is formed by the edges $(f, \phi_i(f))$ for all feasible functions f to which the operator ϕ_i is applicable. This graph is just an A_n-crystal.

Theorem 2.1 [2, Th. 5.1] *For each n and $c \in \mathbb{Z}_+^n$, the n-colored graph $\mathcal{K}(c)$ is exactly the A_n-crystal $K(c)$.*

2.4 Principal lattice and $(n-1)$-colored subcrystals of an A_n-crystal

Based on the crossing model, [2] reveals some important ingredients and relations for an A_n-crystal $K = K(c)$. One of them is the so-called principal lattice, which is defined as follows.

Let $a \in \mathbb{Z}_+^n$ and $a \le c$. One easily checks that the function on the vertices of the supporting graph G that takes the constant value a_k within each subgraph G^k of G, $k = 1, \ldots, n$, is feasible. We denote this function and the vertex of K corresponding to it by $f[a]$ and $\check{v}[a]$, respectively, and call them *principal*. So the set of principal vertices is bijective to the integer box $\mathcal{B}(c) := \{a \in \mathbb{Z}^n \colon \mathbf{0} \le a \le c\}$; this set is called

the *principal lattice* of K and denoted by $\Pi = \Pi(c)$. When it is not confusing, the term "principal lattice" may also be applied to $\mathcal{B}(c)$.

The following properties of the principal lattice will be essentially used later.

Proposition 2.2 [2, Expression (6.4)] *Let $a \in \mathcal{B}(c)$, $k \in \{1, \ldots, n\}$, and $a' := a + 1_k$ (where 1_k is i-th unit base vector in $\mathbb{R}^n$). The principal vertex $\check{v}[a']$ is obtained from $\check{v}[a]$ by applying the operator string*

$$S_{n,k} := w_{n,k,n-k+1} \cdots w_{n,k,2} w_{n,k,1}, \tag{2.6}$$

where for $j = 1, \ldots, n - k + 1$, the substring $w_{n,k,j}$ is defined as

$$w_{n,k,j} := F_j F_{j+1} \cdots F_{j+k-1}.$$

When acting on Π, any two (applicable) strings $S_{n,k}, S_{n,k'}$ commute. In particular, any principal vertex $\check{v}[a]$ is expressed via the source $s_K = \check{v}[\mathbf{0}]$ as

$$\check{v}[a] = S_{n,n}^{a_n} S_{n,n-1}^{a_{n-1}} \cdots S_{n,1}^{a_1}(s_K). \tag{2.7}$$

Proposition 2.3 [2, Prop. 6.1] *For $c', c'' \in Z_+^n$ with $c' \le c'' \le c$, let $K(c' : c'')$ be the subgraph of $K(c)$ formed by the vertices and edges contained in (directed) paths from $\check{v}[c']$ to $\check{v}[c'']$ (the interval of $K(c)$ from $\check{v}[c']$ to $\check{v}[c'']$). Then $K(c' : c'')$ is isomorphic to the A_n-crystal $K(c'' - c')$, and the principal lattice of K' consists of the principal vertices $\check{v}[a]$ of $K(c)$ with $c' \le a \le c''$.*

Let $\mathcal{K}^{(-n)}(c)$ denote the set of subcrystals with colors $1, \ldots, n-1$, and $\mathcal{K}^{(-1)}$ the set of subcrystals with colors $2, \ldots, n$ in K (recall that a subcrystal is assumed to be connected and maximal).

Proposition 2.4 [2, Prop. 7.1] *Each subcrystal in $\mathcal{K}^{(-n)}$ (in $\mathcal{K}^{(-1)}$) contains precisely one principal vertex. This gives a bijection between $\mathcal{K}^{(-n)}$ and Π (resp., between $\mathcal{K}^{(-1)}$ and Π).*

We refer to the members of $\mathcal{K}^{(-n)}$ and $\mathcal{K}^{(-1)}$ as *upper* and *lower* ($(n-1)$-colored) subcrystals of K, respectively. For $a \in \mathcal{B}(c)$, the upper subcrystal containing the vertex $\check{v}[a]$ is denoted by $K^{\uparrow}[a]$. This subcrystal has its own principal lattice of dimension $n - 1$, which is denoted by $\Pi^{\uparrow}[a]$. We say that the coordinate tuple a is the *locus* of $K^{\uparrow}[a]$ (and of $\Pi^{\uparrow}[a]$) in Π. Analogously, for $b \in \mathcal{B}(c)$, the lower subcrystal containing $\check{v}[b]$ is denoted by $K^{\downarrow}[b]$, and its principal lattice by $\Pi^{\downarrow}[b]$; we say that b is the locus of $K^{\downarrow}[b]$ (and of $\Pi^{\downarrow}[b]$) in Π. It turns out that the parameters of upper and lower subcrystals can be expressed explicitly, as follows.

Proposition 2.5 [2, Props. 7.2,7.3] *For $a \in \mathcal{B}(c)$, the upper subcrystal $K^{\uparrow}[a]$ is isomorphic to the A_{n-1}-crystal $K(c^{\uparrow})$, where $c^{\uparrow}$ is the tuple $(c_1^{\uparrow}, \ldots, c_{n-1}^{\uparrow})$ defined by*

$$c_i^{\uparrow} := c_i - a_i + a_{i+1}, \qquad i = 1, \ldots, n-1. \tag{2.8}$$

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The principal vertex $\check{v}[a]$ is contained in the upper lattice $\Pi^{\uparrow}[a]$ and its coordinate $\hbar^{\uparrow} = (\hbar_1^{\uparrow}, \ldots, \hbar_{n-1}^{\uparrow})$ in $\Pi^{\uparrow}[a]$ satisfies

$$\hbar_i^{\uparrow} = a_{i+1}, \qquad i = 1, \ldots, n-1. \tag{2.9}$$

Symmetrically, for $b \in \mathcal{B}(c)$, the lower subcrystal $K^{\downarrow}[b]$ is isomorphic to the A_{n-1}-crystal $K(c^{\downarrow})$ with colors $2, \ldots, n$, where $c^{\downarrow}$ is defined by

$$c_i^{\downarrow} := c_i - b_i + b_{i-1}, \qquad i = 2, \ldots, n. \tag{2.10}$$

The principal vertex $\check{v}[b]$ is contained in the lower lattice $\Pi^{\downarrow}[b]$ and its coordinate $\hbar^{\downarrow} = (\hbar_2^{\downarrow}, \ldots, \hbar_n^{\downarrow})$ in $\Pi^{\downarrow}[b]$ satisfies

$$\hbar_i^{\downarrow} = b_{i-1}, \qquad i = 2, \ldots, n. \tag{2.11}$$

We call $\check{v}[a]$ the *heart* of $K^{\uparrow}[a]$ w.r.t. K, and similarly for lower subcrystals.

3   Assembling an A_n-crystal

As mentioned in the Introduction, the structure of an A_n-crystal $K = K(c)$ will be described in a recursive manner. The idea is as follows. We know that K contains $|\Pi| = (c_1 + 1) \times \ldots \times (c_n + 1)$ upper subcrystals (with colors $1, \ldots, n-1$) and $|\Pi|$ lower subcrystals (with colors $2, \ldots, n$). Moreover, the parameters of these subcrystals are expressed explicitly by (2.8) and (2.10). So we may assume by recursion that the set $\mathcal{K}^{(-n)}$ of upper subcrystals and the set $\mathcal{K}^{(-1)}$ of lower subcrystals are available (already constructed). In order to assemble K, it suffices to characterize, in appropriate terms, the intersection $K^{\uparrow}[a] \cap K^{\downarrow}[b]$ for all pairs $a, b \in \mathcal{B}(c)$ (the intersection may either be empty, or consist of one or more $(n-2)$-colored subcrystals with colors $2, \ldots, n-1$ in K). We give an appropriate characterization in Theorem 3.1 below.

To state it, we need additional terminology and notation. Consider a subcrystal $K^{\uparrow}[a]$, and let $c^{\uparrow}, \hbar^{\uparrow}$ be defined as in (2.8),(2.9). For $p = (p_1, \ldots, p_{n-1}) \in \mathcal{B}(c^{\uparrow})$, the vertex in the upper lattice $\Pi^{\uparrow}[a]$ having the coordinate p is denoted by $v^{\uparrow}[a, p]$. We call the vector $\Delta := p - \hbar^{\uparrow}$ the *deviation* of $v^{\uparrow}[a, p]$ from the heart $\check{v}[a]$ in $\Pi^{\uparrow}[a]$, and will use the alternative notation $v^{\uparrow}[a | \Delta]$ for this vertex. In particular, $\check{v}[a] = v^{\uparrow}[a, \hbar^{\uparrow}] = v^{\uparrow}[a | 0]$.

Similarly, for a lower subcrystal $K^{\downarrow}[b]$, let $c^{\downarrow}, \hbar^{\downarrow}$ be as in (2.10),(2.11). For $q = (q_2, \ldots, q_n) \in \mathcal{B}(c^{\downarrow})$, the vertex with the coordinate q in $\Pi^{\downarrow}[b]$ is denoted by $v^{\downarrow}[a, q]$. Its deviation is $\nabla := q - \hbar^{\downarrow}$, and we may alternatively denote this vertex by $v^{\downarrow}[b | \nabla]$.

We call an $(n-2)$-colored subcrystals with colors $2, \ldots, n-1$ in K a *middle subcrystals* and denote the set of these by $\mathcal{K}^{(-1, -n)}$. Each middle crystal $K^{\uparrow\downarrow}$ is a *lower* subcrystal of some upper subcrystal $K' = K^{\uparrow}[a]$ of K. By Proposition 2.4 applied to K', $K^{\uparrow\downarrow}$ has a unique vertex $v^{\uparrow}[a | \Delta]$ in the lattice $\Pi^{\uparrow}[a]$. So each $K^{\uparrow\downarrow}$ can be encoded by a pair (a, Δ) formed by a locus $a \in \mathcal{B}(c)$ and a deviation Δ in $\Pi^{\uparrow}[a]$. At the same time, $K^{\uparrow\downarrow}$ is an *upper* subcrystal of some lower subcrystal $K^{\downarrow}[b]$

of K and has a unique vertex $v^{\downarrow}[b|\nabla]$ in $\Pi^{\downarrow}[b]$. Therefore, the members of $\mathcal{K}^{(-1,-n)}$ determine a bijection

$$\zeta : (a, \Delta) \mapsto (b, \nabla)$$

between all pairs (a, Δ) concerning upper subcrystals and all pairs (b, ∇) concerning lower subcrystals.

The map ζ is expressed explicitly in the following theorem. Here for a tuple $\rho = (\rho_i : i \in I)$, we denote by ρ^+ (ρ^-) the tuple with the entries $\rho_i^+ := \max\{0, \rho_i\}$ (resp. $\rho_i^- := \min\{0, \rho_i\}$), $i \in I$.

Theorem 3.1 *Let $a \in \mathcal{B}(c)$ and let $\Delta = (\Delta_1, \ldots, \Delta_{n-1})$ be a deviation in $\Pi^{\uparrow}[a]$. Let $(b, \nabla) = \zeta(a, \Delta)$. Then b satisfies*

$$b_i = a_i + \Delta_i^+ + \Delta_{i-1}^-, \qquad i = 1, \ldots, n, \tag{3.1}$$

letting $\Delta_0 = \Delta_n := 0$, and ∇ satisfies

$$\nabla_i = -\Delta_{i-1}, \qquad i = 2, \ldots, n. \tag{3.2}$$

A proof of this theorem will be given in the next section.

Based on Theorem 3.1, the crystal $K(c)$ is assembled as follows. By recursion we assume that all upper and lower subcrystals are already constructed. We also assume that for each upper subcrystal $K^{\uparrow}[a] = K(c^{\uparrow})$, its principal lattice is distinguished by use of the corresponding injective map $\sigma : \mathcal{B}(c^{\uparrow}) \to V(K(c^{\uparrow}))$, and similarly for the lower subcrystals. We delete the edges with color 1 in each $K(c^{\uparrow})$ and extract the components of the resulting graphs, forming a list $\widetilde{\mathcal{K}}$ of all middle subcrystals of $K(c)$. Each $K^{\uparrow\downarrow} \in \widetilde{\mathcal{K}}$ is encoded by a corresponding pair (a, Δ), where $a \in \mathcal{B}(c)$ and the deviation Δ in $\Pi^{\uparrow}[a]$ is determined by use of σ as above. Acting similarly for the lower subcrystals $K(c^{\downarrow})$ (by deleting the edges with color n there), we obtain an isomorphic list of middle subcrystals, each of which being encoded by a corresponding pair (b, ∇), where $b \in \mathcal{B}(c)$ and ∇ is a deviation in $\Pi^{\downarrow}(b)$. Relations (3.1) and (3.2) indicate how to identify each member of the first list with its counterpart in the second one. Now restoring the deleted edges with colors 1 and n, we obtain the desired crystal $K(c)$. The corresponding map $\mathcal{B}(c) \to V(K(c))$ is constructed easily (e.g., by use of operator strings as in Proposition 2.2).

We conclude this section with several remarks.

Remark 1. (3.1) and (3.2) lead to the following expression of a via b and ∇:

$$a_i = b_i + \nabla_i^+ + \nabla_{i+1}^-, \qquad i = 1, \ldots, n, \tag{3.3}$$

letting $\nabla_1 = \nabla_{n+1} := 0$. This will be used, in particular, in the Appendix.

Remark 2. For each $a \in \mathcal{B}(c)$ and each vertex $v = v^{\uparrow}[a, p]$ in the upper lattice $\Pi^{\uparrow}[a]$, one can express the parameter $c^{\uparrow\downarrow} = (c_2^{\uparrow\downarrow}, \ldots, c_{n-1}^{\uparrow\downarrow})$ of the middle subcrystal $K^{\uparrow\downarrow}$ containing v, as well as the coordinate $\hbar^{\uparrow\downarrow} = (\hbar_2^{\uparrow\downarrow}, \ldots, \hbar_{n-1}^{\uparrow\downarrow})$ of its heart w.r.t.

$K^\uparrow[a]$ in the principal lattice of $K^{\uparrow\downarrow}$. Indeed, since $K^{\uparrow\downarrow}$ is a lower subcrystal of $K^\uparrow[a]$, one can apply relations as in (2.10),(2.11). Denoting the parameter of $K^\uparrow[a]$ by $c^\uparrow$ and the coordinate of its heart in $\Pi^\uparrow[a]$ by $\hbar^\uparrow$, letting $\Delta := p - \hbar^\uparrow$, and using (2.8),(2.9), we have:

$$c_i^{\uparrow\downarrow} = c_i^\uparrow - p_i + p_{i-1} = (c_i - a_i + a_{i+1}) - (a_{i+1} + \Delta_i) + (a_i + \Delta_{i-1}) \qquad (3.4)$$
$$= c_i - \Delta_i + \Delta_{i-1}, \qquad i = 2, \ldots, n-1;$$

$$\hbar_i^{\uparrow\downarrow} = p_{i-1} = \hbar_{i-1}^\uparrow + \Delta_{i-1} = a_i + \Delta_{i-1}, \qquad i = 2, \ldots, n-1. \qquad (3.5)$$

Symmetrically, if $K^{\uparrow\downarrow}$ is contained in $K^\downarrow[b]$ and is related to a deviation ∇ in $\Pi^\downarrow[b]$, then

$$c_i^{\uparrow\downarrow} = c_i - \nabla_i + \nabla_{i+1}, \qquad (3.6)$$
$$\hbar_i^{\downarrow\uparrow} = b_i + \nabla_{i+1}, \qquad i = 2, \ldots, n-1, \qquad (3.7)$$

where $\hbar^{\downarrow\uparrow}$ is the coordinate of the heart of $K^{\uparrow\downarrow}$ w.r.t. $K^\downarrow[b]$ in the principal lattice of $K^{\uparrow\downarrow}$ (note that $\hbar^{\downarrow\uparrow}$ may differ from $\hbar^{\uparrow\downarrow}$). We will use formulas (3.4)–(3.7) in subsequent sections.

Remark 3. A straightforward implementation of the above recursive method of constructing $K = K(c)$ takes $O(2^{q(n)}N)$ time and space, where $q(n)$ is a polynomial in n and N is the number of vertices of K. Here the factor $2^{q(n)}$ appears because the total number of vertices in the upper and lower subcrystals is $2N$ (implying that there appear $4N$ vertices in total on the previous step of the recursion, and so on). Therefore, such an implementation has polynomial complexity of the size of the output for each fixed n, but not in general. However, many intermediate subcrystals arising during the recursive process are repeated, and we can use this fact to improve the implementation. More precisely, the colors occurring in each intermediate subcrystal in the process form an interval of the ordered set $(1, \ldots, n)$. We call a subcrystal of this sort a *color-interval subcrystal*, or a *CI-subcrystal*, of K. In fact, *every* CI-subcrystal of K appears in the process. Since the number of intervals is $\frac{n(n+1)}{2}$ and the CI-subcrystals concerning one and the same interval are pairwise disjoint, the total number of vertices of all CI-subcrystals of K is $O(n^2N)$. It is not difficult to implement the recursive process in such a way that each CI-subcrystal K' is explicitly constructed only once. (For this purpose, one can use pointers from the vertices of K' to its source $s_{K'}$ and characterize K' by its color-interval and $s_{K'}$.) As a result, we obtain the following

Proposition 3.2 *Let $n \in \mathbb{Z}_+$ and $c \in \mathbb{Z}_+^n$. The A_n-crystal $K(c)$ and all its CI-subcrystals can be constructed in $O(q'(n)|V(K(c))|)$ time and space, where $q'(n)$ is a polynomial in n.*

Remark 4. Relation (3.1) shows that the intersection of $K^\uparrow[a]$ and $K^\downarrow[b]$ may consist of many middle subcrystals. Indeed, if $\Delta_i > 0$ and $\Delta_{i-1} < 0$ for some i, then b does

not change by simultaneously decreasing Δ_i by 1 and increasing Δ_{i-1} by 1. The number of common middle subcrystals of $K^\uparrow[a]$ and $K^\downarrow[b]$ for arbitrary $a, b \in \mathcal{B}(c)$ can be expressed by an explicit piecewise linear formula, using (3.1) and the box constraints $-a_{i+1} \le \Delta_i \le c_i - a_i$, $i = 1, \ldots, n-1$, on the deviations Δ in $\Pi^\uparrow[a]$ (which follow from (2.8),(2.9)).

4   Proof of Theorem 3.1

Let a, Δ, b, ∇ be as in the hypotheses of this theorem. First we prove relation (3.2) in the assumption that (3.1) is valid.

Proof of (3.2). The middle subcrystal $K^{\uparrow\downarrow}$ determined by (a, Δ) is the same as the one determined by (b, ∇). The parameter $c^{\uparrow\downarrow}$ of $K^{\uparrow\downarrow}$ is expressed simultaneously by (3.4) and by (3.6). Then $c_i - \Delta_i + \Delta_{i-1} = c_i - \nabla_i + \nabla_{i+1}$ for $i = 2, \ldots, n-1$. Therefore,

$$\Delta_1 + \nabla_2 = \Delta_2 + \nabla_3 = \ldots = \Delta_{n-1} + \nabla_n =: \alpha. \tag{4.1}$$

In order to obtain (3.2), one has to show that $\alpha = 0$. We argue as follows. Renumber the colors $1, \ldots, n$ as $n, \ldots, 1$, respectively; this yields the crystal $\widehat{K} = K(\widehat{c})$ symmetric to $K(c)$. Then $K^\downarrow[b]$ turns into the upper subcrystal $\widehat{K}^\uparrow[\widehat{b}]$ of $\widehat{K}$, where $(\widehat{b}_1, \ldots, \widehat{b}_n) = (b_n, \ldots, b_1)$. Also the deviation ∇ in $\Pi^\downarrow[b]$ turns into the deviation $\widehat{\nabla} = (\widehat{\nabla}_1, \ldots, \widehat{\nabla}_{n-1}) = (\nabla_n, \ldots, \nabla_2)$ in the principal lattice of $\widehat{K}^\uparrow[\widehat{b}]$. Applying relations as in (3.1) to $(\widehat{b}, \widehat{\nabla})$, we have

$$\widehat{a}_i = \widehat{b}_i + \widehat{\nabla}_i^+ + \widehat{\nabla}_{i-1}^- = b_{n-i+1} + \nabla_{n-i+1}^+ + \nabla_{n-i+2}^-, \qquad i = 1, \ldots, n, \tag{4.2}$$

where $\widehat{a}_i := a_{n-i+1}$ and $\widehat{\nabla}_n^+ := \widehat{\nabla}_0^- := 0$. On the other hand, (3.1) for (a, Δ) gives

$$\widehat{b}_i = b_{n-i+1} = a_{n-i+1} + \Delta_{n-i+1}^+ + \Delta_{n-i}^-, \qquad i = 1, \ldots, n. \tag{4.3}$$

Relations (4.2) and (4.3) imply

$$a_{n-i+1} = (a_{n-i+1} + \Delta_{n-i+1}^+ + \Delta_{n-i}^-) + \nabla_{n-i+1}^+ + \nabla_{n-i+2}^-,$$

whence

$$\Delta_{n-i+1}^+ + \Delta_{n-i}^- + \nabla_{n-i+1}^+ + \nabla_{n-i+2}^- = 0, \qquad i = 1, \ldots, n.$$

Adding up the latter equalities, we obtain

$$(\Delta_1 + \ldots + \Delta_{n-1}) + (\nabla_2 + \ldots + \nabla_n) = 0.$$

This and (4.1) imply $(n-1)\alpha = 0$. Hence $\alpha = 0$, as required.   ∎

Proof of (3.1). This proof is rather technical and essentially uses the crossing model.

For a feasible function $f \in \mathcal{F}(c)$ and its corresponding vertex v in $K = K(c)$, we may denote v as v_f, and f as f_v. The following observation from the crossing model will be of use:

(4.4) if a vertex $v \in V(K)$ belongs to $K^\uparrow[a]$ and to $K^\downarrow[b]$, then the tuples a and b are expressed via the values of $f = f_v$ in levels n and 1 as follows:

$$a_k = f(v_n^k(n - k + 1)) \text{ and } b_k = f(v_1^k(1)) \text{ for } k = 1, \dots, n.$$

Indeed, the principal vertex $\check{v}[a]$ is reachable from v by applying operators F_i or F_i^{-1} with $i \neq n$. The corresponding moves in the crossing model do not change f within level n. Similarly, $\check{v}[b]$ is reachable from v by applying operators F_i or F_i^{-1} with $i \neq 1$, and the corresponding moves in the crossing model do not change f within level 1. Also the relations in (4.4) are valid for the principal function $f = f_{\check{v}[a]}$.

Next we introduce special functions on the node set $V(G)$ of the supporting graph $G = G_n$. Consider a component $G^k = (V^k, E^k)$ of G. It is a rectangular grid (rotated by $45°$ in the visualization of G), and its vertex set is

$$V^k = \{v_i^k(j) \colon j = 1, \dots, n - k + 1, \ \ i = j, \dots, j + k - 1\}.$$

To represent it in a more convenient form, introduce the variable $m := i - j + 1$ and rename $v_i^k(j)$ as $u_{i-j+1}^k(j)$, or as $u_{i-j+1}(j)$ (when no confusion can arise). Then

$$V^k = \{u_m(j) \colon j = 1, \dots, n - k + 1, \ \ m = 1, \dots, k\},$$

the (descending) SE-edges in G^k are of the form $(u_m(j), u_m(j+1))$, and the (ascending) NE-edges are of the form $(u_m(j), u_{m-1}(j))$. We specify the following subsets of V^k:
   (i) the *SW-side* $P = P^k := \{u_k(1), \dots, u_k(n - k + 1)\}$;
   (ii) the *right rectangle* $R = R^k := \{u_m(j) \colon 1 \leq m \leq k - 1, \ 1 \leq j \leq n - k + 1\}$;
   (iii) the *left rectangle* $L = L^k := \{u_m(j) \colon 1 \leq m \leq k, \ 1 \leq j \leq n - k\}$.
Denote the characteristic functions (in $\mathbb{R}^{V^k}$) of P, R, L as π^k, ρ^k, λ^k, respectively.

Return to $a \in \mathcal{B}(c)$ and a deviation Δ in $\Pi^\uparrow[a]$. Associate to (a, Δ) the functions

$$f_{a,\Delta}^k := a_k \pi^k + (a_k + \Delta_{k-1}^-)\rho^k + \Delta_k^+ \lambda^k \tag{4.5}$$

on V^k for $k = 1, \dots, n$ (see Fig. 1), and their direct sum

$$f_{a,\Delta} := f_{a,\Delta}^1 \oplus \dots \oplus f_{a,\Delta}^n$$

(the function on $V(G)$ whose restriction to each V^k is $f_{a,\Delta}^k$).

In view of (4.4), $f = f_{a,\Delta}$ takes the values in levels n and 1 as required in (3.1) (with k in place of i), namely, $f(v_n^k(n - k + 1)) = a_k$ and $f(v_1^k(1)) = a_k + \Delta_k^+ + \Delta_{k-1}^-$ for $k = 1, \dots, n$. Therefore, to obtain (3.1) it suffices to show the following

Lemma 4.1 (i) *The function* $f = f_{a,\Delta}$ *is feasible.* (ii) *The vertex* v_f *is the vertex of* $\Pi^\uparrow[a]$ *having the deviation* Δ.

Proof First we prove statement (i). Let $k \in \{1, \dots, n\}$. We partition V^k into four subsets (rectangular *pieces*):

$$Z_1^k := P^k \cap L^k; \quad Z_2^k := L^k \setminus P^k; \quad Z_3^k := \{u_k^k(n - k + 1)\}; \quad Z_4^k := R^k \setminus L^k$$

(where $Z_2^k = Z_4^k = \emptyset$ when $k = 1$, and $Z_1^k = Z_2^k = \emptyset$ when $k = n$). By (4.5),
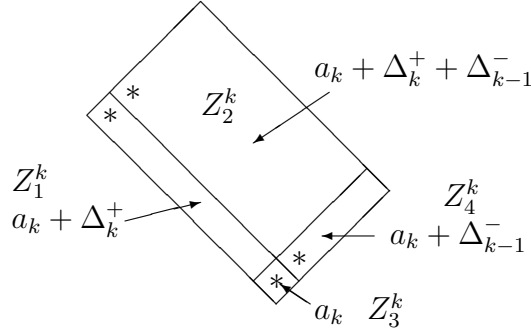
Figure 1: The partition of V^k.

(4.6) f takes a constant value within each piece Z_q^k, namely: $a_k + \Delta_k^+$ on Z_1^k;
  $a_k + \Delta_k^+ + \Delta_{k-1}^-$ on Z_2^k; a_k on Z_3^k; and $a_k + \Delta_{k-1}^-$ on Z_4^k

(as illustrated in Fig. 1). Also each edge of G'^k connecting different pieces goes either from Z_1^k to $Z_2^k \cup Z_3^k$ or from $Z_2^k \cup Z_3^k$ to Z_4^k. This and (4.6) imply that $\partial f(e) \geq 0$ for each edge $e \in E^k$, whence f satisfies (2.1)(i).

The deviation Δ is bounded as $-\hbar^\uparrow \leq \Delta \leq c^\uparrow - \hbar^\uparrow$, where $c^\uparrow$ is the parameter of the subcrystal $K^\uparrow[a]$ and $\hbar^\uparrow$ is the coordinate of its heart $\check{v}[a]$ in $\Pi^\uparrow[a]$. Expressions (2.8) and (2.9) for $c^\uparrow$ and $\hbar^\uparrow$ give

$$-a_{k+1} \leq \Delta_k \leq c_k - a_k \quad \text{and} \quad -a_k \leq \Delta_{k-1} \leq c_{k-1} - a_{k-1}. \tag{4.7}$$

The inequalities $\Delta_k \leq c_k - a_k$ and $a_k \leq c_k$ imply $a_k + \Delta_k^+ \leq c_k$. And the inequalities $-a_k \leq \Delta_{k-1}$ and $a_k \geq 0$ imply $a_k + \Delta_{k-1}^- \geq 0$. Then, in view of (4.6), we obtain $0 \leq f(v) \leq c_k$ for each node v of G'^k, yielding (2.1)(ii).

To verify the switch condition (2.1)(iii), consider a multinode $V_i(j)$ with $i < n$. It consists of $n - i + 1$ nodes $v_i^k(j)$, where $i - j + 1 \leq k \leq n - j + 1$.

Let $i \leq n - 2$. Suppose that there is a node $v = v_i^k(j)$ whose SW-edge $e = (u, v)$ exists and is not f-tight. This is possible only if $u \in Z_1^k$ and $v \in Z_2^k$. Then k is determined as $k = i - j + 2$, i.e. v is the second node in $V_i(j)$. We observe that: (a) for the first node $v_i^{k-1}(j)$ of $V_i(j)$, both ends of its SE-edge e' belong to the piece Z_1^{k-1}, whence e' is f-tight; and (b) for any node $v_i^{k'}(j)$ with $k' > k$ in $V_i(j)$, both ends of its SW-edge e'' belong either to $Z_2^{k'}$ or to $Z_4^{k'}$. Therefore, the node v satisfies the requirement in (2.1)(iii) for $V_i(j)$.

Now let $i = n - 1$. Then $V_i(j)$ consists of two nodes $v = v_{n-1}^{n-j}(j)$ and $v' = v_{n-1}^{n-j+1}(j)$. Put $k := n - j$. Then the edge $e = e^{\mathrm{SE}}(v)$ goes from Z_1^k to $Z_3^k = \{u_k^k(n-k+1)\}$, and the edge $e' = e^{\mathrm{SW}}(v')$ goes from $Z_3^{k+1} = \{u_{k+1}^{k+1}(n-k)\}$ to Z_4^{k+1}. By (4.6), we have $\partial f(e) = (a_k + \Delta_k^+) - a_k = \Delta_k^+$ and $\partial f(e') = a_{k+1} - (a_{k+1} + \Delta_k^-) = -\Delta_k^-$. Since at least one of Δ_k^+, Δ_k^- is zero, we conclude that at least one of e, e' is tight. So (2.1)(iii) is valid again.

Next we start proving statement (ii) in the lemma. We use induction on the value

$$\eta(\Delta) := \Delta_1 + \ldots + \Delta_{n-1}.$$

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In view of (4.7), $\eta(\Delta) \geq -a_2 - \ldots - a_n$. Let this hold with equality. Then $\Delta_k = -a_{k+1} \leq 0$ for $k = 1, \ldots, n-1$, and by (4.6), $f = f_{a,\Delta}$ takes the following values within each V^k: $f(v) = a_k$ if $v \in P^k$, and $f(v) = 0$ if $v \in V^k - P^k$. This f is the minimal feasible function whose values in level n match a, and therefore, v_f is the source of $K^\uparrow[a]$. Then v_f is the minimal vertex $\check{v}[a, \mathbf{0}]$ in $\Pi^\uparrow[a]$, and its deviation in $\Pi^\uparrow[a]$ is just Δ, as required. This gives the base of the induction.

Now consider an arbitrary Δ satisfying (4.7). Let k be such that $\Delta_k < c_k - a_k$ (if any) and define $\Delta'_k := \Delta_k + 1$ and $\Delta'_i := \Delta_i$ for $i \neq k$. Then $\eta(\Delta) < \eta(\Delta')$. We assume by induction that claim (ii) is valid for $f_{a,\Delta}$, and our aim is to show validity of (ii) for $f_{a,\Delta'}$.

In what follows f stands for the initial function $f_{a,\Delta}$.

Let v' be the vertex with the deviation Δ' in $\Pi^\uparrow[a]$. Both v_f and v' are principal vertices of the subcrystal $K^\uparrow[a]$ and the coordinate of v' in $\Pi^\uparrow[a]$ is obtained from the one of v_f by increasing its k-th entry by 1. According to Proposition 2.2 (with n replaced by $n-1$), v' is obtained from v_f by applying the operator string

$$S_{n-1,k} = w_{n-1,k,n-k} \cdots w_{n-1,k,1},$$

where $w_{n-1,k,j} = F_j \cdots F_{j+k-1}$ (cf. (2.6)). In light of this, we have to show that

(4.8) when (the sequence of moves corresponding to) $S_{n-1,k}$ is applied to f, the resulting feasible function is exactly $f_{a,\Delta'}$.

For convenience m-th term, from left to right, in the substring $w_{n-1,k,j}$ (i.e. the operator F_{j+m-1}) will be denoted by $\phi(j, m)$, $m = 1, \ldots, k$. So $w_{n-1,k,j} = \phi(j, 1)\phi(j, 2)\ldots\phi(j, k)$.

We distinguish between two cases: $\Delta \geq 0$ and $\Delta < 0$.

Case 1: $\Delta_k \geq 0$. An essential fact is that the number $k(n-k)$ of operators in $S_{n-1,k}$ is equal to the number of nodes in the left rectangle L^k of G^k, and moreover, the substrings in $S_{n-1,k}$ one-to-one correspond to the NE-paths in L^k. More precisely, the level of each node $u_m(j)$ of L^k is equal to the "color" of the operator $\phi(j, m)$ (indeed, $u_m(j) = v^k_{j+m-1}(j)$ and $\phi(j, m) = F_{j+m-1}$).

Let $f^{j,m}$ denote the current function on $V(G)$ just before the application of $\phi(j, m)$ (when the process starts with $f = f_{a,\Delta}$). Also we write $(j', m') \prec (j, m)$ if $j' < j$ or if $j' = j$ and $m' > m$. We assert that

(4.9) for each m, the application of $\phi(j, m)$ to $f^{j,m}$ increases the value at the node $u^k_m(j)$ by 1; equivalently: $f^{j,m}(u^k_{m'}(j')) = f(u^k_{m'}(j')) + 1$ if $(j', m') \prec (j, m)$, and $f^{j,m}(v) = f(v)$ for the other nodes v of G,

whence (4.8) will immediately follow.

In order to show (4.9), we first examine tight edges and the slacks $\varepsilon(v)$ of the nodes v in levels $< n$ for the initial function f. One can observe from (4.6) that

(4.10) for $k' = 1, \ldots, n$, each node v of the subgraph $G^{k'}$ has at least one entering edge (i.e. $e^{\mathrm{SW}}(v)$ or $e^{\mathrm{NW}}(v)$) which is f-tight, except, possibly, for the nodes $v_{k'}^{k'}(1)$, $v_{k'-1}^{k'}(1)$, $v_n^{k'}(n-k'+1)$, $v_{n-1}^{k'}(n-k'+1)$ (indicated by stars in Fig. 1).

Claim. *For $k' = 1, \ldots, n$ and a node v of $G^{k'}$ in a level $< n$,*
   (a) if $v \neq v_{k'}^{k'}(1), v_{k'-1}^{k'}(1)$, then $\varepsilon(v) = 0$;
   (b) if $v = v_{k'}^{k'}(1)$, then $\varepsilon(v) = c_{k'} - a_{k'} - \Delta_{k'}^+ \geq 0$;
   (c) if $v = v_{k'-1}^{k'}(1)$, then $\varepsilon(v) = -\Delta_{k'}^- \geq 0$.

Proof of Claim. Let $v = v_i^{k'}(j)$ and $i < n$. By (2.2), the slack $\varepsilon(v)$ is equal to $f(w) + f(z) - f(u) - f(v)$, where $w := v_{i-1}^{k'}(j-1)$, $z := v_{i+1}^{k'}(j)$, $u := v_i^{k'}(j-1)$ (these vertices belong to the extended graph $\bar{G}^{k'}$). We consider possible cases and use (4.6).
   (i) If w, z, u are in $G^{k'}$, then $\partial f(w, v) = \partial f(u, z)$.
   (ii) If both v, w are in the piece $Z_1^{k'}$ of $G^{k'}$, then $f(w) = f(v)$ and $f(u) = f(z) = c_{k'}$.
   (iii) If $j = 1$ and $i \leq k - 2$, then $f(v) = f(z)$ and $f(u) = f(w) = c_{k'}$. So in these cases we have $\varepsilon(v) = 0$, yielding (a).
   (iv) Let $v = v_{k'}^{k'}(1)$. Then $f(v) = a_{k'} + \Delta_{k'}^+$ and $f(u) = f(w) = f(z) = c_{k'}$. This gives $\varepsilon(v) = c_{k'} - a_{k'} - \Delta_{k'}^+$, yielding (b).
   (v) Let $v = v_{k'-1}^{k'}(1)$. Then $f(v) = a_{k'} + \Delta_{k'}^+ + \Delta_{k'-1}^-$, $f(z) = a_{k'} + \Delta_{k'}^+$ and $f(u) = f(w) = c_{k'}$. This gives $\varepsilon(v) = -\Delta_{k'-1}^-$, yielding (c). ∎

This Claim and the relations $\Delta_k^- = 0$ and $\Delta_k < c_k - a_k$ enable us to estimate the total slacks $\varepsilon_i(j)$ for f at the multinodes $V_i(j)$ with $i < n$:

(4.11) (i) the edge $e^{\mathrm{SW}}(v_k^{k+1}(1))$ is f-tight, $\varepsilon(v_k^k(1)) > 0$, and $\varepsilon(v) = 0$ for the other nodes v in $V_k(1)$; so $\varepsilon_k(1) > 0$;

   (ii) if $i \neq k, n$, then $\varepsilon(v_i^i(1)), \varepsilon(v_i^{i+1}(1)) \geq 0$ and $\varepsilon(v) = 0$ for the other nodes v in $V_i(1)$; so $\varepsilon_i(1) \geq 0$;

   (iii) if $i \neq n$ and $j > 1$, then $\varepsilon(v) = 0$ for all nodes v in $V_i(j)$; so $\varepsilon_i(j) = 0$.

Now we are ready to prove (4.9). When dealing with a current function $f^{j,m}$ and seeking for the node at level $j + m - 1$ where the operator $\phi(j, m)$ should act to increase $f^{j,m}$, we can immediately exclude from consideration any node v that has a tight entering edge (since acting the operator at v would cause violation of the monotonicity condition (2.1)(i)).

Due to (4.10) and (4.11)(i), for the initial function $f = f^{1,k}$, there is only one node in level k that has no tight entering edge, namely, $v_k^k(1)$. So, at the first step of the process, the first operator $\phi(1, k)$ of $S_{n-1,k}$ acts just at $v_k^k(1)$, as required in (4.9).

Next consider a step with $f' := f^{j,m}$ and $\phi(j, m)$ for $(j, m) \neq (1, k)$, assuming that (4.9) is valid at the previous step.

(A) Let $j = 1$ (and $m < k$). For $v := v_m^k(1)$ and $z := v_{m+1}^k(1)$, we have $f'(v) = f(v) \leq f(z) = f'(z) - 1$. So the unique edge $e = (z, v)$ entering v is not f'-tight. By (4.10), there are at most two other nodes in level m that may have no tight entering edges for f (and therefore, for f'), namely, $v_m^m(1)$ and $v_m^{m+1}(1)$. Then $\phi(1, m)$

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must act at v, as required in (4.9) (since the non-tightness of the SW-edge e of v implies that none of the nodes $v_m^{m'}(1)$ in $V_m(1)$ preceding v (i.e. with $m' < k$) can be the switch-node).

(B) Let $j > 1$. Comparing f' with f in the node $v := u_m^k(j) = v_{j+m-1}^k(j)$ and its adjacent nodes, we observe that v has no f'-tight entering edge and that $\varepsilon_{f'}(v) > 0$. Also for any other node v' in level $j + m - 1$, one can see that if v' has a tight entering edge for f, then so does for f', and that $\varepsilon_f(v') \geq \varepsilon_{f'}(v') \geq 0$. Using this, properties $(4.10), (4.11)$(iii), and condition (2.5), one can conclude that the total and reduced slacks for f' at the multinode $V' := V_{j+m-1}(j)$ are positive, that V' is the active multinode for f' in level $j + m - 1$, and that $\phi(j, m)$ can be applied only at v, yielding (4.9) again.

Thus, (4.8) is valid in Case 1.

Case 2: $\Delta_k < 0$. We assert that in this case the string $S_{n-1,k}$ acts within the right rectangle R^{k+1} of the subgraph G^{k+1} (note that R^{k+1} is of size $k \times (n - k)$). More precisely,

(4.12) each operator $\phi(j, m)$ modifies the current function by increasing its value at the node $u_m^{k+1}(j)$ by 1.

Then for the resulting function $\widetilde{f}$ in the process, its restriction to V^{k+1} is

$$a_{k+1}\pi^{k+1} + (a_{k+1} + \Delta_k^- + 1)\rho^{k+1} + \Delta_{k+1}^+ \lambda^{k+1}$$

(cf. (4.5)). Therefore, $\widetilde{f} = f_{a,\Delta'}$ (in view of $(\Delta')_k^- = \Delta_k^- + 1$), yielding (4.8).

To show (4.12), we argue as in the previous case and use (4.10) and the above Claim. Since $\Delta_k < 0$, part (i) in (4.11) for the initial function f is modified as:

(4.13) for $j = 1, \ldots, n - k$, the SW-edge of each node $u_k^{k+1}(j) = v_{j+k-1}^{k+1}(j)$ is not f-tight, $\varepsilon(v_k^{k+1}(1)) > 0$, $\varepsilon(v_k^k(1)) \geq 0$, and $\varepsilon(v) = 0$ for the other nodes v in $V_k(1)$; so $\varepsilon_k(1) > 0$,

while properties (ii) and (iii) preserve.

By (4.10) and (4.13), there are only two nodes in level k that have no f-tight entering edges, namely, $v_k^k(1)$ and $v_k^{k+1}(1)$. Also $e = e^{\mathrm{SW}}(v_k^{k+1}(1))$ is not tight. So, at the first step, $\phi(1, k)$ must act at $v_k^{k+1}(1)$, as required in (4.12) (since the non-tightness of e implies that the node $v_k^k(1)$ preceding $v_k^{k+1}(1)$ cannot be the switch-node in $V_k(1)$).

The fact that $\phi(1, m)$ with $m < k$ acts at $v_m^{k+1}(1)$ is shown by arguing as in (A) above. And for $j > 1$, to show that $\phi(j, m) = F_{j+m-1}$ acts at $u_m^{k+1}(j) = v_{j+m-1}^{k+1}(j)$, we argue as in (B) above. Here, when $m = k$, we also use the fact that the edge $e^{\mathrm{SW}}(u_k^{k+1}(j))$ is not f-tight (by (4.13)), whence both edges entering $u_k^{k+1}(j)$ are not tight for the current function. So (4.12) is always valid.

Thus, we have the desired property (4.8) in both cases 1 and 2, and statement (ii) in the lemma follows. ∎

This completes the proof of relation (3.1) in Theorem 3.1.

5 Illustrations

In this section we give two illustrations to the above assembling construction for A-crystals. The first one specifies the interrelation between upper and lower subcrystals in an arbitrary A_2-crystal, which can be compared with the explicit construction (the so-called "sail model") for A_2-crystals in [1]. The second one visualizes the subcrystals structure for one instance of A_3-crystals, namely, $K(1,1,1)$.

5.1 A_2-crystals

The subcrystals structure becomes simpler when we deal with an A_2-crystal $K = K(c_1, c_2)$. In this case the roles of upper, lower, and middle subcrystals are played by 1-paths, 2-paths, and vertices of K, respectively, where by an i-path we mean a maximal path of color i.

Consider an upper subcrystal in K. This is a 1-path $P = (v_0, v_1, \ldots, v_p)$ containing exactly one principal vertex $\check{v}[a]$ of K (the heart of P); here v_i stands for i-th vertex in P, $a = (a_1, a_2) \in \mathbb{Z}_+^2$ and $a \leq c$. Let $\check{v}[a] = v_h$. Formulas (2.8) and (2.9) give

$$|P| = p = c_1 - a_1 + a_2 \qquad \text{and} \qquad h = a_2. \tag{5.1}$$

Fix a vertex $v = v_i$ of P. It belongs to some 2-path (lower subcrystal) $Q = (u_1, u_2, \ldots, u_q)$. Let $v = u_j$ and let $\check{v}[b] = u_{\bar{h}}$ be the principal vertex of K occurring in Q (the heart of Q). The vertex v forms a middle subcrystal of K; its deviations from the heart of P and from the heart of Q are equal to $i - h =: \delta$ and $j - \bar{h} =: \bar{\delta}$, respectively. By (3.2) in Theorem 3.1, we have $\bar{\delta} = -\delta$. Then we can compute the coordinates b by use of (3.1) and, further, apply (2.10) and (2.11) to compute the length of Q and the locus of its heart. This gives:

(5.2) (i) if $\delta \geq 0$ (i.e. $a_2 \leq i \leq c_1 - a_1 + a_2$), then $b_1 = a_1 + \delta = a_1 + i - a_2$, $b_2 = a_2$, $|Q| = c_2 - b_2 + b_1 = c_2 - 2a_2 + a_1 + i$, and $|Q| - \bar{h} = |Q| - b_1 = c_2 - a_2$;

(ii) if $\delta \leq 0$ (i.e. $0 \leq i \leq a_2$), then $b_1 = a_1$, $b_2 = a_2 + \delta = a_2 + (i - a_2) = i$, $|Q| = c_2 - b_2 + b_1 = c_2 - i + a_1$, and $\bar{h} = b_1 = a_1$.

Using (5.1) and (5.2), one can enumerate the sets of 1-paths and 2-paths and properly intersect corresponding pairs, obtaining the A_2-crystal $K(c)$. It is rather routine to check that the resulting graph coincides with the one generated by the *sail model* from [1]. Next we outline that construction (it will be used in Section 9.1).

Given $c \in \mathbb{Z}_+^2$, the A_2-crystal $K(c)$ is produced from two particular two-colored graphs R and L, called the *right sail* of size c_1 and the *left sail* of size c_2, respectively. The vertices of R correspond to the vectors $(i, j) \in \mathbb{Z}^2$ such that $0 \leq j \leq i \leq c_1$, and the vertices of L to the vectors $(i, j) \in \mathbb{Z}^2$ such that $0 \leq i \leq j \leq c_2$. In both R, L, the edges of color 1 are all possible pairs of the form $((i, j), (i + 1, j))$, and the edges of color 2 are all possible pairs of the form $((i, j), (i, j + 1))$. (Observe that both R and L satisfy axioms (A1)-(A4), R is isomorphic to $K(c_1, 0)$, L is isomorphic to $K(0, c_2)$, and their critical vertices are the "diagonal vertices" (i, i).)

In order to produce $K(c)$, take c_2 disjoint copies $R_1, \ldots, R_{c_2}$ of R and c_1 disjoint copies $L_1, \ldots, L_{c_1}$ of L, referring to R_j as j-th right sail, and to L_i as i-th left sail. Let $D(R_j)$ and $D(L_i)$ denote the sets of diagonal vertices in R_j and L_i, respectively. For all $i = 1, \ldots, c_1$ and $j = 1, \ldots, c_2$, we identify the diagonal vertices $(i, i) \in D(R_j)$ and $(j, j) \in D(L_i)$. The resulting graph is just the desired $K(c)$. The edge colors of $K(c)$ are inherited from L and R. One checks that $K(c)$ has $(c_1 + 1) \times (c_2 + 1)$ critical vertices; they are exactly those induced by the diagonal vertices of the sails. The principal lattice of $K(c)$ is just constituted by the critical vertices.

The case $(c_1, c_2) = (1, 2)$ is drawn in the picture; here the critical (principal) vertices are indicated by circles, 1-edges by horizontal arrows, and 2-edges by vertical arrows.
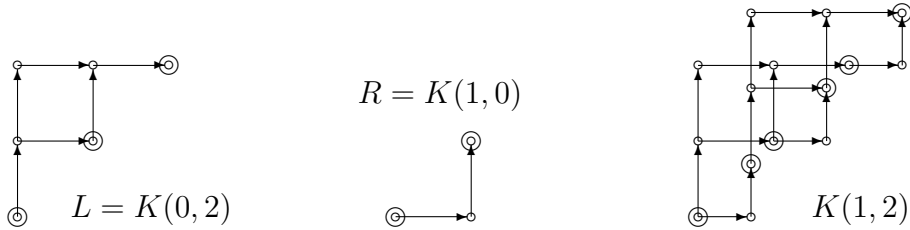
In particular, the sail model shows that the numbers of edges of each color in an A_2-crystal are the same. This implies a similar property for any A_n-crystal (and moreover, for crystals of classical simply-laced types).

5.2 A_3-crystal $K(1, 1, 1)$

Next we illustrate the A_3-crystal $K = K(1, 1, 1)$. It has 64 vertices and 102 edges, and drawing it in full would take too much space; for this reason, we describe it in fragments, namely, by demonstrating all of its upper and lower subcrystals. We abbreviate notation $\check{v}[(i, j, k)]$ for principal vertices to (i, j, k) for short. So the principal lattice consists of eight vertices $(0, 0, 0), \ldots, (1, 1, 1)$, as drawn in the picture (where the arrows indicate moves by principal operator strings $S_{3,k}$ as in (2.6)):

$$S_{3,1} = F_3 F_2 F_1$$
$$S_{3,2} = F_2 F_3 F_1 F_2$$
$$S_{3,3} = F_1 F_2 F_3$$
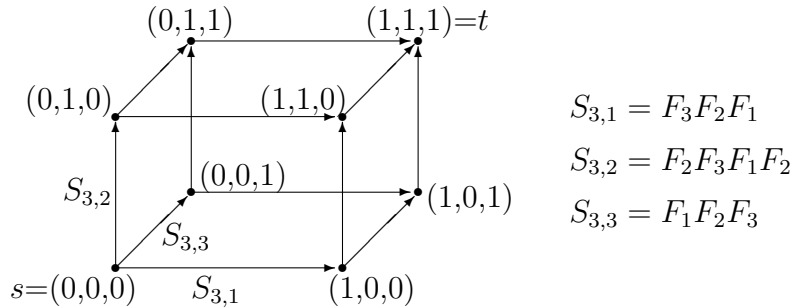
Thus, K has eight upper subcrystals $K^\uparrow[i, j, k]$ and eight lower subcrystals $K^\downarrow[i, j, k]$ (writing $K^\bullet[i, j, k]$ for $K^\bullet[(i, j, k)]$); they are drawn in Figures 2 and 3.

Here the directions of edges of colors 1,2,3 are as indicated in the upper left corner. In each subcrystal we indicate its critical vertices by black circles, and the unique principal vertex of K occurring in it (the heart) by a big white circle. K has 30 middle subcrystals (paths of color 2), which are labeled as $A, \ldots, Z, \Gamma, \Delta, \Phi, \Psi$ (note that $B, F, G, N, P, T, V, \Phi$ consist of single vertices).
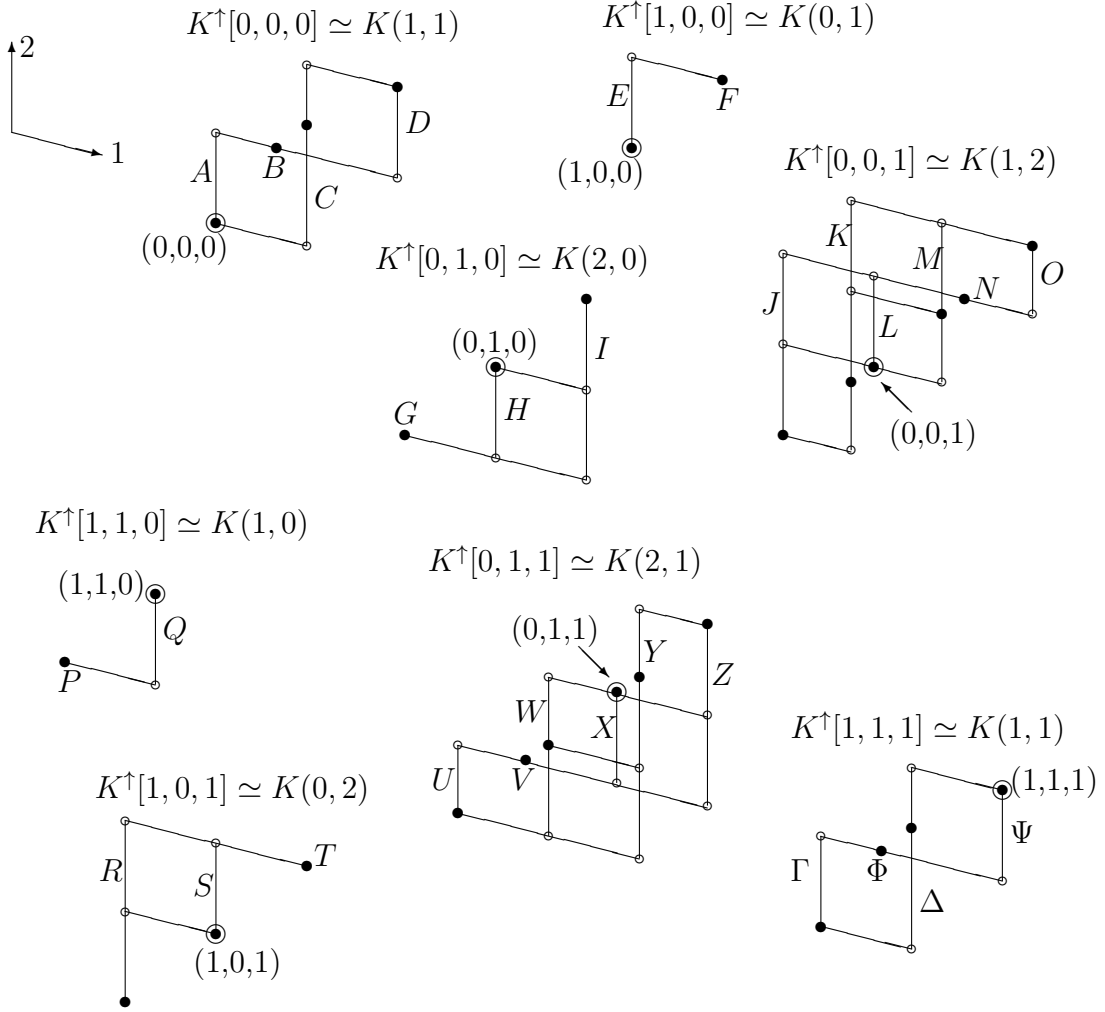
Figure 2: The upper subcrystals in $K(1,1,1)$

For each upper subcrystal $K^\uparrow[i, j, k]$, its parameter $c^\uparrow$ and heart locus $\hbar^\uparrow$, computed by (2.8) and (2.9), are as follows (where $\widetilde{K}, \widetilde{s}, z$ denote the current subcrystal, its source, and its heart, respectively):

- for $K^\uparrow[0, 0, 0]$: $c_1^\uparrow = 1 - 0 + 0 = 1$, $c_2^\uparrow = 1 - 0 + 0 = 1$, and $\hbar_1^\uparrow = \hbar_2^\uparrow = 0$ (so $\widetilde{K}$ is isomorphic to $K(1,1)$ and z coincides with $\widetilde{s}$);
- for $K^\uparrow[1, 0, 0]$: $c_1^\uparrow = 1 - 1 + 0 = 0$, $c_2^\uparrow = 1 - 0 + 0 = 1$, and $\hbar_1^\uparrow = \hbar_2^\uparrow = 0$;

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- for $K^{\uparrow}[0,1,0]$: $c_1^{\uparrow} = 1 - 0 + 1 = 2$, $c_2^{\uparrow} = 1 - 1 + 0 = 0$, $\hbar_1^{\uparrow} = 1$, and $\hbar_2^{\uparrow} = 0$ (so $\widetilde{K} \simeq K(2,0)$ and z is located at $S_{2,1}(\widetilde{s}) = F_2 F_1(\widetilde{s})$);
- for $K^{\uparrow}[0,0,1]$: $c_1^{\uparrow} = 1 - 0 + 0 = 1$, $c_2^{\uparrow} = 1 - 0 + 1 = 2$, $\hbar_1^{\uparrow} = 0$, and $\hbar_2^{\uparrow} = 1$ (so $\widetilde{K} \simeq K(1,2)$ and z is located at $S_{2,2}(\widetilde{s}) = F_1 F_2(\widetilde{s})$);
- for $K^{\uparrow}[1,1,0]$: $c_1^{\uparrow} = 1 - 1 + 1 = 1$, $c_2^{\uparrow} = 1 - 1 + 0 = 0$, $\hbar_1^{\uparrow} = 1$, and $\hbar_2^{\uparrow} = 0$;
- for $K^{\uparrow}[1,0,1]$: $c_1^{\uparrow} = 1 - 1 + 0 = 0$, $c_2^{\uparrow} = 1 - 0 + 1 = 2$, $\hbar_1^{\uparrow} = 0$, and $\hbar_2^{\uparrow} = 1$;
- for $K^{\uparrow}[0,1,1]$: $c_1^{\uparrow} = 1 - 0 + 1 = 2$, $c_2^{\uparrow} = 1 - 1 + 1 = 1$, and $\hbar_1^{\uparrow} = \hbar_2^{\uparrow} = 1$ (so $\widetilde{K} \simeq K(2,1)$ and z is located at $S_{2,2}S_{2,1}(\widetilde{s}) = F_1 F_2 F_2 F_1(\widetilde{s})$);
- for $K^{\uparrow}[1,1,1]$: $c_1^{\uparrow} = 1 - 1 + 1 = 1$, $c_2^{\uparrow} = 1 - 1 + 1 = 1$, and $\hbar_1^{\uparrow} = \hbar_2^{\uparrow} = 1$.
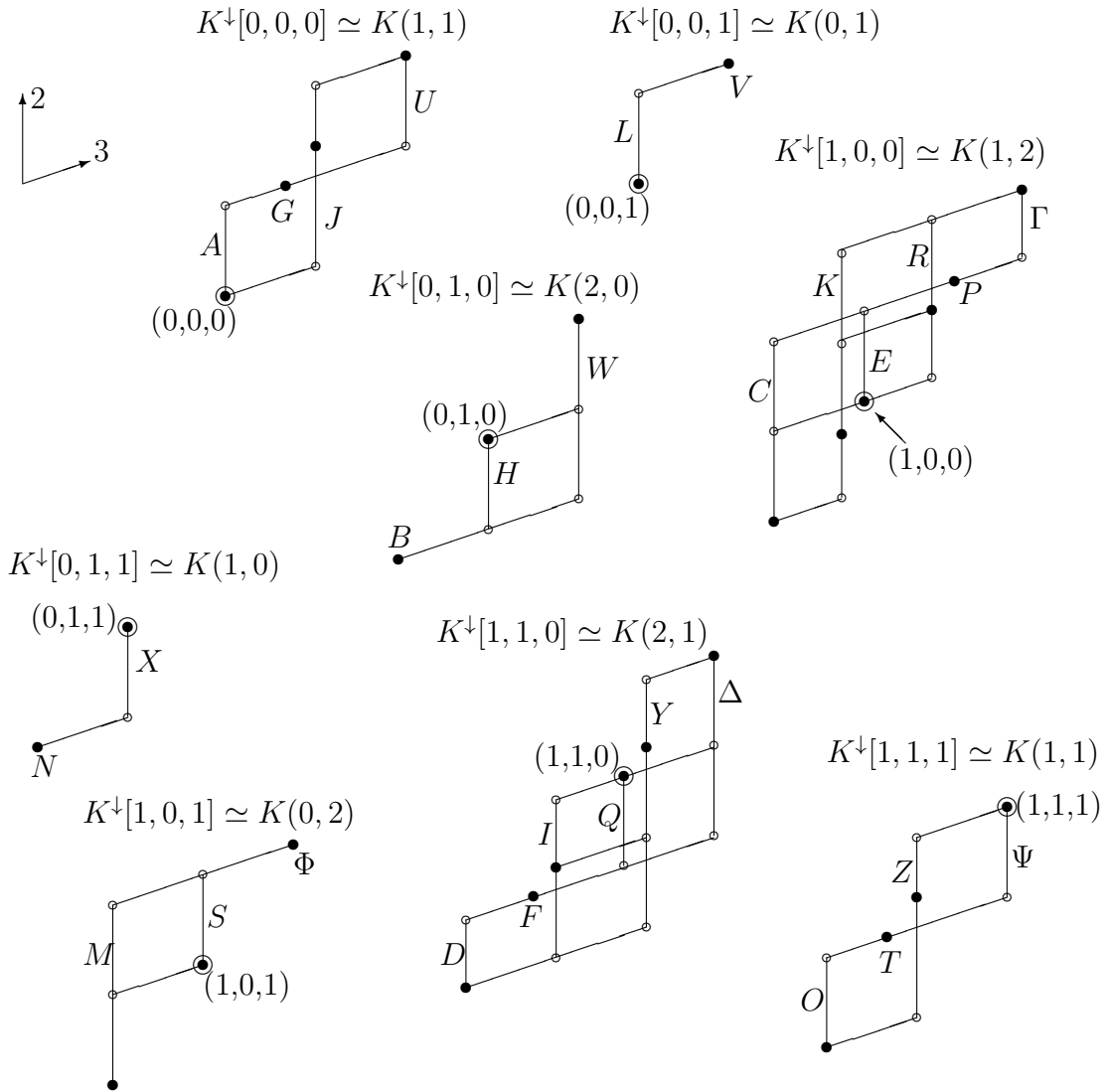

Figure 3: The lower subcrystals in $K(1,1,1)$

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Since $K(1,1,1)$ is "symmetric", so are its upper and lower subcrystals, i.e. each $K^{\downarrow}[i,j,k]$ is obtained from $K^{\uparrow}[k,j,i]$ by replacing color 1 by 3. In Fig. 3, when writing $K^{\downarrow}[i,j,k] \simeq K(\alpha,\beta)$, the parameters α,β concern colors 3 and 2, respectively.

Now the desired $K(1,1,1)$ is assembled by gluing the fragments in Figs. 2,3 along the 2-paths $A,\dots,\Psi$.

6   Deriving B_n-crystals from symmetric A_{2n-1}-crystals

We say that an A_{2n-1}-crystal $K = (V(K), E_1 \sqcup \dots \sqcup E_{2n-1})$ with parameter $c = (c_1, \dots, c_{2n-1})$ is *symmetric* if $c_i = c_{2n-i}$ for each i. Equivalently: renumbering the colors $1, \dots, 2n-1$ as $2n-1, \dots, 1$ makes the same K (since any A-crystal is determined by its parameter). Color $2n-i$ is regarded as *complementary* to color i and will usually be denoted with prime: we write i' for $2n-i$. In particular, $n' = n$. For an operator string $F_{i_1}^{\alpha_1} F_{i_2}^{\alpha_2} \dots F_{i_k}^{\alpha_k}$ (where each F_{i_j} concerns the color class E_{i_j} and $\alpha_j \in \mathbb{Z}$), the complementary string is defined to be $F_{i_1'}^{\alpha_1} F_{i_2'}^{\alpha_2} \dots F_{i_k'}^{\alpha_k}$. This gives a natural complementarity relation on the set of paths, including non-directed ones, that begin at the source s of K, which in turn yields the complementarity bijection (involution) $\sigma : V(K) \to V(K)$. We extend σ, in a natural way, to edges, paths and subgraphs of K. A vertex $v \in V(K)$ is called *self-complementary* (or *symmetric*) if $v = \sigma(v)$; equivalently: for some (equivalently, *any*) path P from s to v, the complementary path $\sigma(P)$ terminates at v as well. In particular, the source and sink of K are self-complementary.

Let S be the set of self-complementary vertices in K. Clearly if a vertex $u \in S$ has outgoing edge colored i, then v has outgoing i'-edge as well. When $i < n$, the colors i, i' are not neighboring ($|i - i'| \geq 2$); so by axiom (A5) (from Section 2.1), the operators F_i and $F_{i'}$ commute at v, and the vertex $v = F_i F_{i'}(u)$ is again self-complementary. We denote such a pair (i, i') with $i < i'$ by $\bar{i}$. The pair (u,v) of vertices as above is regarded as an *edge with color $\bar{i}$*, or an *$\bar{i}$-edge*; we denote the set of $\bar{i}$-edges by $E_{\bar{i}}$ and denote the partial operator on S related to $E_{\bar{i}}$ by $F_{\bar{i}}$. We also refer to the four edges $(u,w), (w,v), (u,w'), (w',v)$ of K, where $w := F_i(u)$ and $w' := F_{i'}(u)$, as the *underlying* edges of (u,v).

As to color n, if $u \in S$ has outgoing n-edge (u,v), then $v \in S$ as well. We formally set $\bar{n} := (n, n' = n)$, define $E_{\bar{n}}$ to be the set of n-edges connecting pairs of self-complementary vertices (so $E_{\bar{n}} \subseteq E_n$), and associate to $E_{\bar{n}}$ the partial operator $F_{\bar{n}}$ on S.

As a result, we obtain the n-colored directed graph $B = (S, E_{\bar{1}} \sqcup \dots \sqcup E_{\bar{n}})$, called the *symmetric extract* from K. The colors in B are ordered as $\bar{1}, \dots, \bar{n}$, and different colors $\bar{i}, \bar{j}$ are called *neighboring* if $|i - j| = 1$.

Figure 4 illustrates two "simplest" symmetric A-crystals for $n = 3$, namely, $K(1,0,1)$ and $K(0,1,0)$, and their symmetric extracts $B(1,0)$ and $B(0,1)$.

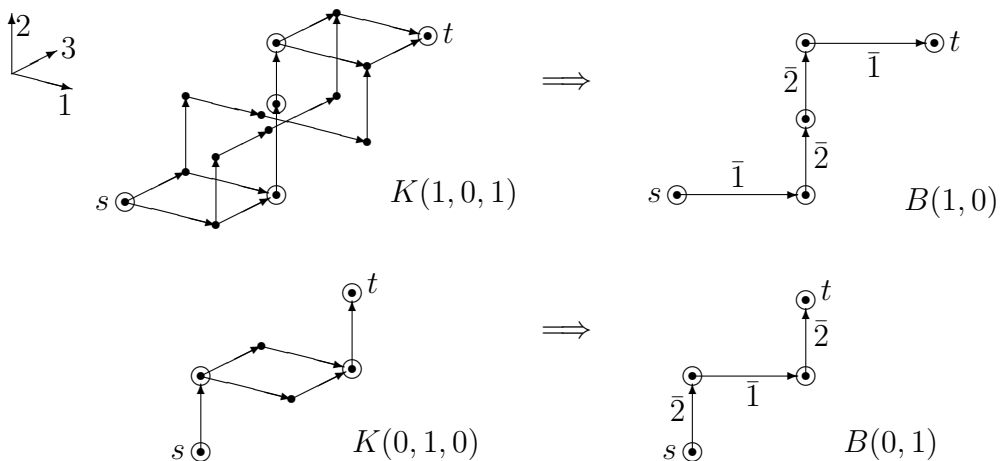The following relation between A- and B-crystals can be concluded from [12, Th. 3.2.4].

Figure 4: Creation of $B(1,0)$ and $B(0,1)$.

Theorem 6.1 *Let $K(c)$ be a symmetric A_{2n-1}-crystal. Then the symmetric extract $B = (S, E_{\bar{1}} \sqcup \ldots \sqcup E_{\bar{n}})$ from $K(c)$ is a B_n-crystal.*

In Sections 6–8 we give a combinatorial proof of this theorem, based on our knowledge of the structure of A_n- and B_2-crystals. We start with simple observations.

1. An important property of $K = K(c)$ is that it is *graded* at each color i, which means that in any closed route in K, the numbers of forward and backward i-edges are equal (equivalently, $V(K)$ admits a map to $\mathbb{Z}^n$ under which each i-edge corresponds to a shift by i-th base vector). Clearly such a property remains valid for B as well. In particular, B is acyclic. Also for $i = 1, \ldots, n$, each vertex of B has at most one outgoing $\bar{i}$-edge and at most one incoming $\bar{i}$-edge. So each component of $(S, E_{\bar{i}})$ is a path (as required in (A1)). Another known fact is that the graph K^{rev} obtained from (a not necessarily symmetric A-crystal) K by reversing its edges and changing each edge color to the complementary one is isomorphic to K (this operation swaps the source and sink). This implies that the "reversed" graph B^{rev} (with preserved edge colors) is isomorphic to B; this fact will be used in Section 8.3.

2. For a vertex $v \in S$ and color $\bar{i}$, let $h_{\bar{i}}(v)$ (resp. $t_{\bar{i}}(v)$) denote the length of the maximal $\bar{i}$-path beginning (resp. ending) at v. Then

$$h_{\bar{i}}(v) = h_i(v) = h_{i'}(v) \quad \text{and} \quad t_{\bar{i}}(v) = t_i(v) = t_{i'}(v). \tag{6.1}$$

This is trivial when $i = n$. If $i < n$, consider the component K' of $(V(K), E_i \sqcup E_{i'})$ that contains v. Since $|i - i'| \geq 2$, K' is the Cartesian product of an i-path P and an i'-path P'. Since K' contains a self-complementary vertex, it easily follows that $K' = \sigma(K')$. This implies that the lengths of P and P' are equal, and further, that $V(K') \cap S$ consists of the vertices of the form $F_i^{\alpha} F_{i'}^{\alpha}(s')$, where s' is the source of K' (the "diagonal" of K'). Now (6.1) easily follows.

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3. Each vertex $v \in S$ is reachable by a (directed) path in B beginning at the source s of K; in particular, B is connected and s is the source of B. This can be shown by induction on the length $|P|$ of a path P from s to v in K. Indeed, for such a P, take the complementary path P' (also going from s to v). Let the last edge (u, v) of P have color i; then the last edge (u', v) of P' has color i'. If $i = n$, then $u = u'$, implying $u \in S$, and we can apply induction. And if $i \neq n$, then F_i^{-1} and $F_{i'}^{-1}$ commute at v and the vertex $w := F_i^{-1} F_{i'}^{-1}(v)$ is self-complementary. Since K is graded, the length of a path from s to w in K is less than $|P|$. So by induction w is reachable by a path from s in B, implying a similar property for v.

4. By (6.1) applied to $v = s$, we have $h_{\bar{i}}(s) = c_i$ for each $i = 1, \ldots, n$. So one can regard the n-tuple $\bar{c} = (c_1, \ldots, c_n)$ as the *parameter* of B and denote B as $B(\bar{c})$ (the set of such tuples $\bar{c}$ is $\mathbb{Z}_+^n$, and there is a unique B for each $\bar{c}$ in our construction).

5. Let $i, j < n$ and $|i - j| \geq 2$. Then any two colors among i, i', j, j' are not neighboring, and therefore (by (A5)), each subcrystal K' of K with these four colors is the Cartesian product of four monochromatic paths. This implies that if two operators among $F_{\bar{i}}, F_{\bar{i}}^{-1}, F_{\bar{j}}, F_{\bar{j}}^{-1}$ act at $v \in S$, then these operators commute at v, whence each component of $(S, E_{\bar{i}} \sqcup E_{\bar{j}})$ is the Cartesian product of an $\bar{i}$-path and a $\bar{j}$-path, i.e. an $A_1 \times A_1$-crystal. A similar fact is shown for $(S, E_{\bar{i}} \sqcup E_{\bar{n}})$ when $i \leq n - 2$. Thus, B satisfies axiom (BC1) (from Section 2.2).

It remains to verify axioms (BC2),(BC3),(BC4) for B. We start with (BC2).

Lemma 6.2 *Let $i, j < n$ and $|i - j| = 1$. Then each component of the subgraph $B' := (S, E_{\bar{i}} \sqcup E_{\bar{j}})$ is an A_2-crystal.*

Proof To verify axiom (A2) for B', consider an $\bar{i}$-edge (u, v). Let $x := F_i(u)$ and $y := F_{i'}(u)$; then $(u, x), (y, v)$ are the i-edges and $(u, y), (x, v)$ are the i'-edges of K underlying (u, v). Since $|i - j| = |i' - j'| = 1$ and $|i - j'| = |i' - j| \geq 2$, we obtain (using (A2),(A5) for K):

$$h_j(v) = h_j(x) = h_j(u) + \ell_j(u, x); \quad h_j(v) = h_j(y) + \ell_j(y, v) = h_j(u) + \ell_j(y, v);$$
$$h_{j'}(v) = h_{j'}(x) + \ell_{j'}(x, v) = h_{j'}(u) + \ell_{j'}(x, v); \quad h_{j'}(v) = h_{j'}(y) = h_{j'}(u) + \ell_{j'}(u, y)$$

(labels ℓ are defined for A-crystals in Section 2.1). These and the equalities $h_j(u) = h_{j'}(u) = h_{\bar{j}}(u)$ and $h_j(v) = h_{j'}(v) = h_{\bar{j}}(v)$ (cf. (6.1)) imply

$$\ell_j(u, x) = \ell_j(y, v) = \ell_{j'}(x, v) = \ell_{j'}(u, y) =: \alpha,$$

and

$$h_{\bar{j}}(v) = h_{\bar{j}}(u) + \alpha.$$

Handling lengths $t_j, t_{j'}$ in a similar way, we obtain $t_{\bar{j}}(v) = t_{\bar{j}}(u) + 1 - \alpha$.

Therefore, when traversing (u, v), the lengths $h_{\bar{j}}$ and $t_{\bar{j}}$ behave as required for A-crystals, and

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(6.2) all underlying edges of an $\bar{i}$-edge e have one and the same label α w.r.t. their neighboring colors in $\{j, j'\}$, and e inherits just this label: $\ell_{\bar{j}}(e) = \alpha$.

To check the convexity condition in (A2), consider consecutive $\bar{i}$-edges $(u, v), (v, w)$ and take their underlying i-edges (x, v) and (v, y'), where $x := F_i^{-1}(v)$ and $y' := F_i(v)$. Then $\ell_j(x, v) \leq \ell_j(v, y')$ (by (A2) for K) implies $\ell_{\bar{j}}(u, v) \leq \ell_{\bar{j}}(v, w)$, by (6.2). Thus, B' satisfies axiom (A2).

Next, let $u \in S$ have outgoing $\bar{i}$-edge (u, v) and outgoing $\bar{j}$-edge (u, v') in B, and let $\ell_{\bar{j}}(u, v) = 0$. By (A2), $h_{\bar{j}}(v) \geq h_{\bar{j}}(u)$; so v has outgoing $\bar{j}$-edge (v, w). Similarly, v' has outgoing $\bar{i}$-edge (v', w'). We assert that $w = w'$ (as required in (A3)), i.e.

$$F_{\bar{j}} F_{\bar{i}}(u) = F_{\bar{i}} F_{\bar{j}}(u).$$

To show this, let $x := F_{i'}(u)$ and $y := F_j(u)$ (these vertices are not in S). Then u, x, y and $z := F_j(x)$ are connected by the i'-edges $(u, x), (y, z)$ and the j-edges $(u, y), (x, z)$ (since $|i' - j| \geq 2$). By (6.2), $\ell_{\bar{j}}(u, v) = 0$ implies $\ell_{j'}(u, x) = 0$. Therefore, $h_{j'}(x) = h_{j'}(u)$. This together with the trivial equalities $h_{j'}(y) = h_{j'}(u)$ and $h_{j'}(z) = h_{j'}(x)$ (as $|j - j'| \geq 2$) implies $h_{j'}(y) = h_{j'}(z)$, which means that $\ell_{j'}(y, z) = 0$. So the operators $F_{i'}, F_{j'}$ commute at y; note that $F_{j'}(y) = v'$. We have

$$w = F_{j'} F_j F_{i'} F_i(u) = F_{j'} F_{i'} F_j F_i(u) = F_{j'} F_{i'} F_i F_j(u)$$
$$= F_{j'} F_i F_{i'} F_j(u) = F_i F_{j'} F_{i'} F_j(u) = F_i F_{i'} F_{j'} F_j(u) = w'$$

(since: F_i, F_j commute at u; $F_{i'}, F_{j'}$ commute at $y = F_j(u)$; and F_p, F_q with $|p - q| \geq 2$ are permutable). Also the fact that $\ell_i(u, y) = 1$ (by (A3) for K) implies $\ell_{\bar{i}}(u, v') = 1$. Thus, B' satisfies the part of (A3) concerning the forward operators $F_{\bar{i}}, F_{\bar{j}}$. The claim for the backward operators $F_{\bar{i}}^{-1}, F_{\bar{j}}^{-1}$ follows by reversing the edges of K and B.

Finally, instead of a direct (and tiresome) verification of axiom (A4) for B', we can appeal to the result in [1, Proposition 5.3] saying that for a connected two-colored graph K', (A4) follows from (A1),(A2),(A3) and the condition that K' has exactly one source (zero-indegree vertex).

In light of this, consider a component $\widetilde{B}$ of B'. Let $\widetilde{K}$ be the component of $(V(K), E_i \sqcup E_{i'} \sqcup E_j \sqcup E_{j'})$ containing the vertices of $\widetilde{B}$. This $\widetilde{K}$ is the Cartesian product of two A_2-crystals (with colors i, j and colors i', j'), whence $\widetilde{K}$ has a unique source $\widetilde{s}$. We claim that $\widetilde{s}$ is the unique source of $\widetilde{B}$.

Indeed, since $\widetilde{B}$ is finite and acyclic, it has a source s'. Suppose $s' \neq \widetilde{s}$. Then s' has an incoming edge e of some color among $\{i, i', j, j'\}$. Let for definiteness e be an i-edge. Then F_i^{-1} acts at s', and by the symmetry, so does $F_{i'}^{-1}$. These operators commute, and $v := F_{i'}^{-1} F_i^{-1}(s')$ is a self-complementary vertex. Then (v, s') is an edge of $\widetilde{B}$ entering s', contrary to the choice of s'. Thus, $s' = \widetilde{s}$, and (A4) for B' follows.

This completes the proof of the lemma. ∎

The crucial point is to show validity of (BC3) and (BC4). Since these axioms concern only colors $n-1, n, n+1$ in K, we may assume that $n=2$. Here we use the simple fact that if a component K' of $(V(K), E_{n-1} \sqcup E_n \sqcup E_{n+1})$ contains a self-complementary vertex, then for each $v \in V(K')$, the vertex $\sigma(v)$ belongs to K' as well. Hence K' is symmetric.

Theorem 6.3 *Let $K = (V(K), E_1 \sqcup E_2 \sqcup E_3)$ be a symmetric A_3-crystal with parameter $c = (c_1, c_2, c_3 = c_1)$. Then the symmetric extract $B(\bar{c}) = (S, E_{\bar{1}} \sqcup E_{\bar{2}})$ from K is the B_2-crystal with parameter $\bar{c} = (c_1, c_2)$ respecting the Cartan coefficients $m_{12} = -2$ and $m_{21} = -1$.*

7   The worm model

Our method of proof of Theorem 6.3 (given in the next section) consists in showing that the graph B figured there is isomorphic to the graph generated by the so-called *worm model* for the given parameter $\bar{c} = (c_1, c_2)$. This relies on the fact that the latter graph is just the B_2-crystal for $\bar{c}$. In this section we review the construction of *worm graphs* and operations on them given in [3].

Given a parameter $\bar{c} = (c_1, c_2) \in \mathbb{Z}_+^2$, the worm model produces a two-colored directed graph $W = W(\bar{c})$, called the *worm graph* for $\bar{c}$. The vertices of W are the admissible six-tuples $w = (x', y, x''; y', x, y'')$ of integers satisfying

$$0 \le x, x', x'' \le 2c_1 \quad \text{and} \quad 0 \le y, y', y'' \le c_2. \tag{7.1}$$

Here the six-tuple w is called *admissible* if the following three conditions hold:

(7.2) (i) x' and x'' are even;
     (ii) $y' \le y \le y''$ and $x' \le x \le x''$;
     (iii) if $y' < y$ then $x' = x$, and if $y < y''$ then $x = x''$.

It is convenient to visualize w by taking four points in the rectangle $R(c_1, c_2) := \{(\alpha, \beta) \in \mathbb{R}^2 : 0 \le \alpha \le 2c_1, \ 0 \le \beta \le c_2\}$, namely:

$$X' = (x', y), \quad X'' = (x'', y), \quad Y' = (x, y') \quad \text{and} \quad Y'' = (x, y''),$$

and drawing the horizontal line-segment $X'X''$ connecting X' and X'' and the vertical line-segment $Y'Y''$ connecting Y' and Y''. Then (7.2) is equivalent to the following:

(7.3) (i) the first coordinates of the points X' and X'' are even;
     (ii) the point X'' lies to the right of X', and the point Y'' lies above Y';
     (iii) the segments $X'X''$ and $Y'Y''$ have nonempty intersection;
     (iv) at least one of $X' = X''$, $X' = Y''$, $Y' = Y''$, $Y' = X''$ holds.

Depending on the equality in (7.3)(iv), we distinguish between four sorts of vertices of W, also called *worms*:

V-worm (viz. *vertical* worm) appears when $Y' \le Y''$ and $X' = X''$;
VH-worm: $Y' \le Y'' = X' \le X''$;
HV-worm: $X' \le X'' = Y' \le Y''$;
H-worm (viz. *horizontal* worm): $X' \le X''$ and $Y' = Y''$.

(When, e.g., $X' = X'' = Y'$, we regard w as a V-worm and an HV-worm simultaneously.) These cases are illustrated (from left to right) in the picture; hereinafter X stands for the point $X' = X''$, and Y for $Y' = Y''$.

$$Y'' \quad\quad Y'' = X' \quad\quad X'' \quad\quad Y'' \quad\quad X' \quad Y \quad X''$$

$$X$$

$$Y' \quad\quad\quad Y' \quad\quad\quad X' \quad X'' = Y'$$
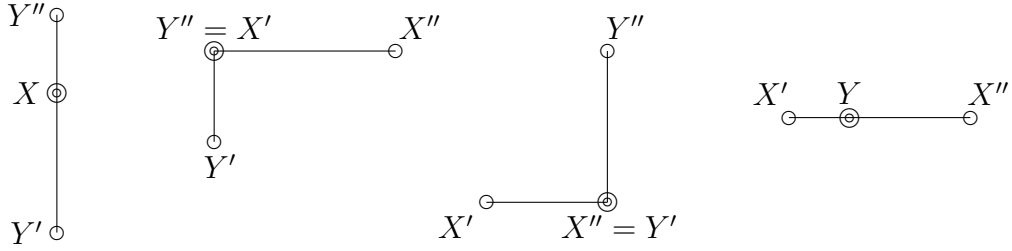
A worm w is called *proper* if three points among X', X'', Y', Y'' are different. When a worm degenerates into one point, we say that the worm is *principal* (which matches a principal vertex in the related B_2-crystal). The horizontal line-segment $X'X''$ is called the *horizontal limb* of w (which degenerates into the single point X in the V-worm case). Also when Y' (resp. Y'') does not lie in the line-segment $X'X''$, we say that the vertical line-segment $Y'X'$ is the *lower limb* (resp. $X''Y''$ is the *upper limb*) of w.

Next we explain the construction of edges of W. We denote the edge colors by $\widetilde{1}$ and $\widetilde{2}$, and write $\widetilde{\mathbf{1}}$ and $\widetilde{\mathbf{2}}$ for the partial operators on the worms associated to these colors, respectively. The action of $\widetilde{\mathbf{1}}$ on a worm $v = (x', y, x''\,;\, y', x, y'')$ is as follows:

(7.4)   (i) if $2x > x' + x''$ then x' increases by 2;
    (ii) if $x = x' = x''$ and $y'' > y$ then y increases by 1;
    (iii) otherwise x'' increases by 2

(preserving the other entries). The operator does not act if the new six-tuple would violate the boundary condition (7.1). So in case of a proper HV-worm, the point X' moves by two positions to the right; in case of a VH-worm, the point X'' moves by two positions to the right; in case of a V-worm with $X \ne Y''$, the point X moves by one position up. The case of H-worms is a bit tricky: one should move (by two positions to the right) that of the points X', X'' which is farther from Y; if they are equidistant from Y, then the point X'' moves.

In its turn, the action of $\widetilde{\mathbf{2}}$ on v is as follows:

(7.5)   (iv) if $2y > y' + y''$, then y' increases by 1;
    (v) if $y'' = y = y'$ and $x'' > x$, then x increases by 1;
    (vi) otherwise y'' increases by 1.

So the operator $\widetilde{\mathbf{2}}$ shifts Y' (Y'') by one position up in the proper VH-case (resp. in the HV-case) and shifts Y by one position to the right in the H-case with $Y \neq X''$. In the V-case, $\widetilde{\mathbf{2}}$ shifts (by one position up) that of the points Y', Y'' which is farther from X; if they are equidistant from X, then Y'' moves.

Theorem 7.1 [3] *For each $\bar{c} \in \mathbb{Z}_+^2$, the worm graph $W(\bar{c})$ is isomorphic to the B_2-crystal $B(\bar{c})$ (where colors $\widetilde{1}, \widetilde{2}$ correspond to $1, 2$, respectively).*

8 Symmetric extracts from A_3-crystals are B_2-crystals

In this section we use the above worm model to prove Theorem 6.3 (thus completing the proof of Theorem 6.1). The proof falls into three stages, described in Sections 8.1–8.3 below.

Let $K = K(c)$ be a symmetric A_3-crystal with parameter $c = (c_1, c_2, c_3 = c_1)$, and $B = B(\bar{c})$ the symmetric extract from K, where $\bar{c} = (c_1, c_2)$. As before, S denotes the set of self-complementary vertices in K, or the vertices of B. For brevity the partial operators F_1, F_2, F_3 on the vertices of K are denoted as $\mathbf{1}, \mathbf{2}, \mathbf{3}$, and the corresponding operators $F_{\bar{1}}, F_{\bar{2}}$ for B by $\bar{\mathbf{1}}, \bar{\mathbf{2}}$ (respectively).

8.1 Additional relations

Our first goal is to establish additional facts (in Lemma 8.1) about self-complementary vertices of K which will be needed to relate B to the worm graph for (c_1, c_2) defined in Section 7.

In case $n = 3$, relations in (2.6) on the principal lattice Π of K are specified as:

$$\begin{aligned} S_{3,1} &= w_{3,1,3}w_{3,1,2}w_{3,1,1} = \mathbf{321}; \\ S_{3,2} &= w_{3,2,2}w_{3,2,1} = \mathbf{2312} = \mathbf{2132}; \\ S_{3,3} &= w_{3,3,1} = \mathbf{123}. \end{aligned}$$

These relations together with (2.7) and the fact that any operator strings $S_{3,k}$ and $S_{3,k'}$ commute within Π imply that for any principal vertex $\check{v}[a = (a_1, a_2, a_3)]$ of K, its complementary vertex $\sigma(\check{v}[a])$ is also principal and has the form $\check{v}[(a_3, a_2, a_1)]$. (To see this, take the path from the source s of K to $\check{v}[a]$ corresponding to the string $S_{3,3}^{a_3}S_{3,2}^{a_2}S_{3,1}^{a_1}$. Its complementary path goes to $\sigma(\check{v}[a])$ and corresponds to the string $S_{3,1}^{a_3}S_{3,2}^{a_2}S_{3,3}^{a_1}$.) Furthermore, the lower subcrystal $K^{\downarrow}[(a_3, a_2, a_1)]$ (with colors 2,3) is complementary to the upper subcrystal $K^{\uparrow}[a]$ (with colors 1,2).

Consider a self-complementary vertex $v \in S$ of K. It belongs to some upper subcrystal $K^{\uparrow}[a]$ and some lower subcrystal $K^{\downarrow}[b]$. The component (middle subcrystal) of $K^{\uparrow}[a] \cap K^{\downarrow}[b]$ containing v is a path P of color 2. By a general fact (cf. Proposition 2.4), P has exactly one vertex, z say, in the upper lattice $\Pi^{\uparrow}[a]$; similarly, P has exactly one vertex, z' say, in the lower lattice $\Pi^{\downarrow}[b]$. Let $\Delta = (\Delta_1, \Delta_2)$ be the deviation of z in $\Pi^{\uparrow}[a]$ (from the heart $\check{v}[a]$ of $K^{\uparrow}[a]$), and $\nabla = (\nabla_2, \nabla_3)$ the deviation

of z' in $\Pi^\downarrow[b]$. Using notation from Section 3, we denote z and z' as $v^\uparrow[a|\Delta]$ and $v^\downarrow[b|\nabla]$, respectively. The next lemma exhibits important features of v.

Lemma 8.1 *For a, b, Δ, ∇ as above, the following properties hold:*
   (i) $(b_1, b_2, b_3) = (a_3, a_2, a_1)$;
   (ii) $K^\downarrow[b]$ *is complementary to* $K^\uparrow[a]$;
   (iii) $\Delta_1 = -\Delta_2 = \nabla_3 = -\nabla_2$;
   (iv) $v^\uparrow[a|\Delta] = v^\downarrow[b|\nabla]$.

Proof We can reach the vertex v from the source s of K by moving along the concatenation of three paths P_1, P_2, P_3, where: P_1 goes from s to $\check{v}[a]$; P_2 is a path from $\check{v}[a]$ to $v^\uparrow[a|\Delta]$ in $K^\uparrow[a]$; and P_3 is a path from $v^\uparrow[a|\Delta]$ to v in P. (The paths P_2, P_3 are not necessarily directed.) Take the complementary paths P_1', P_2', P_3' to P_1, P_2, P_3, respectively.

The vertex v belongs to the lower subcrystal $K^\downarrow[b]$. On the other hand, v belongs to the lower subcrystal K' complementary to $K^\uparrow[a]$, which is expressed as $K^\downarrow[(a_3, a_2, a_1)]$ (since the end of P_1 is the principle vertex $\check{v}[a]$ of K and the end of P_1' is the complementary principle vertex $\sigma(\check{v}[a]) = \check{v}[(a_3, a_2, a_1)]$). This yields (i),(ii).

By relation (3.2) in Theorem 3.1, we have $\nabla_2 = -\Delta_1$ and $\nabla_3 = -\Delta_2$. Also since the end $v^\downarrow[b|\nabla]$ of P_2' is complementary to the end $v^\uparrow[a|\Delta]$ of P_2, the deviation ∇ is complementary to Δ, i.e. $\nabla_2 = \Delta_2$ and $\nabla_3 = \Delta_1$. This yields (iii).

Finally, since the monochromatic paths P_3 and P_3' have the same color 2 and end at the same vertex v, we have $P_3' = P_3$. Therefore, the beginning vertices $v^\uparrow[a|\Delta]$ and $v^\downarrow[b|\nabla]$ of these paths coincide, yielding (iv). ∎

8.2 A correspondence between symmetric vertices and worms

In this subsection we explain how to associate the elements of S to the vertices (worms) of the worm graph $W(\bar{c})$.

For $v \in S$ and its corresponding a, Δ as above, we will write $a(v)$ for a, and $\Delta(v)$ for Δ. By (iii) in Lemma 8.1, the deviation Δ is of the form

$$\Delta = (\Delta_1, \Delta_2) = (\delta, -\delta)$$

for some $\delta = \delta(v) \in \mathbb{Z}$. By (i) in that lemma, $b_1 = a_3$; this together with the equality $b_1 = a_1 + \Delta_1^+$ (cf. (3.1) in Theorem 3.1) implies $a_3 = a_1 + \Delta_1^+$. This gives certain constrains on a, δ, namely:

(8.1) one always holds $a_3 \geq a_1$; furthermore, if $a_3 > a_1$ then $\delta = a_3 - a_1 \; (> 0)$, and if $a_3 = a_1$ then $\delta \leq 0$.

So $a(v), \Delta(v)$ are determined by $a_1(v), a_2(v), \delta(v)$. The latter triple together with the coordinate $\ell = \ell(v)$ of v in the corresponding middle subcrystal (path of color 2) $P = P(v)$ determine v in S. We will refer to the quadruple $(a_1(v), a_2(v), \delta(v), \ell(v))$ as

the *description* of v. In the monochromatic subcrystal P, the principal lattice consists of all vertices of P; the heart is the vertex $v^\uparrow[a|\Delta]$, further denoted by $z = z(v)$; and ℓ is the length of the subpath in P from the beginning to v (the *tail length* for v).

The description (a_1, a_2, δ, ℓ) of v satisfies the following linear constrains:

$$0 \le a_1 \le c_1; \qquad 0 \le a_2 \le c_2; \tag{8.2}$$
$$-a_2, -c_2 + a_2 \le \delta \le c_1 - a_1; \tag{8.3}$$
$$0 \le \ell \le c_2 + 2\delta. \tag{8.4}$$

Here (8.2) is clear. When $\delta \ge 0$, the right inequality in (8.3) follows from $a_1 + \delta = a_3 \le c_1$; cf. (8.1). The upper subcrystal $K^\uparrow[a]$ has parameter $c^\uparrow$ with

$$c_1^\uparrow = c_1 - a_1 + a_2, \qquad c_2^\uparrow = c_2 - a_2 + a_3 \tag{8.5}$$

(by (2.8)) and heart coordinate $\hbar^\uparrow$ with

$$\hbar_1^\uparrow = a_2, \qquad \hbar_2^\uparrow = a_3 \tag{8.6}$$

(by (2.9)). When $\delta \le 0$, we obtain $-\delta = -\Delta_1 \le \hbar_1^\uparrow = a_2$ and $-\delta = \Delta_2 \le c_2^\uparrow - \hbar_2^\uparrow = c_2 - a_2$, yielding the left inequalities in (8.3). Finally, for the middle subcrystal (path) P, its parameter (length) $c_2^{\uparrow\downarrow}$ and coordinate $\hbar_2^{\uparrow\downarrow}$ of its heart z are:

$$c_2^{\uparrow\downarrow} = c_2 - \Delta_2 + \Delta_1 = c_2 + 2\delta, \qquad \hbar_2^{\uparrow\downarrow} = a_2 + \Delta_1 = a_2 + \delta \tag{8.7}$$

(by (3.4) and (3.5)). The first relation in (8.7) yields (8.4).

Conversely, let a_1, a_2, δ, ℓ be integers satisfying (8.2)–(8.4). Put $a_3 := a_1 + \delta^+$, $a := (a_1, a_2, a_3)$ and $b := (a_3, a_2, a_1)$. Comparing (8.2),(8.3) with (8.5),(8.6), one can conclude that the upper lattice $\Pi^\uparrow[a]$ contains a vertex whose deviation equals $(\delta, -\delta)$. Moreover, this vertex $v^\uparrow[a|(\delta, -\delta)] =: z$ coincides with the vertex $v^\downarrow[b|(-\delta, \delta)]$ in the lower lattice $\Pi^\downarrow[b]$ and is self-complementary. Then all vertices of the middle subcrystal P containing z are self-complementary. Now comparing (8.4) with (8.7), we can conclude that P has a vertex v whose coordinate equals ℓ. This v is just the self-complementary vertex corresponding to a_1, a_2, δ, ℓ.

Thus, we obtain the following

Theorem 8.2 *For each integer solution (a_1, a_2, δ, ℓ) to (8.2)–(8.4), there is a vertex $v \in S$ such that $(a_1, a_2, \delta, \ell) = (a_1(v), a_2(v), \delta(v), \ell(v))$, and vice versa.* ∎

This correspondence is crucial in our construction of worms for the elements of S. It is convenient to consider the prism $\{a \in \mathbb{R}^3 : 0 \le a_1 \le a_3 \le c_1, \ 0 \le a_2 \le c_2\}$; the integer points in it are exactly the coordinates of principal vertices $\check{v}[a]$ of K with $a_1 \le a_3$. The ground rectangle for the worms that we construct is identified with the facet Φ of the prism formed by the points a satisfying $a_1 = a_3$. We modify the coordinates on Φ by $(a_1, a_2, a_3 = a_1) \mapsto (2a_1, a_2)$; then the first coordinate runs from 0 to $2c_1$, and the second from 0 to c_2.

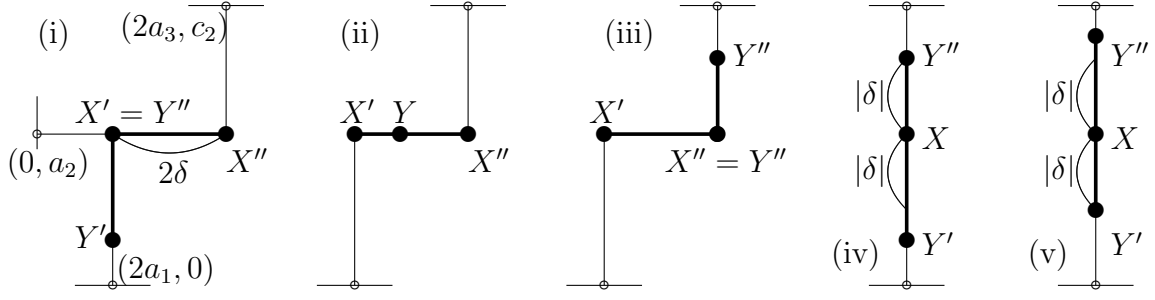Figure 5: (i) $\delta > 0$, $\ell < a_2$; (ii) $\delta > 0$, $a_2 < \ell < a_2 + 2\delta$; (iii) $\delta > 0$, $\ell > a_2 + 2\delta$; (iv) $\delta < 0$, $\ell < a_2 + \delta$; (v) $\delta < 0$, $\ell > a_2 + \delta$.

Consider $v \in S$ and let a, δ, ℓ stand for $a(v), \delta(v), \ell(v)$, respectively. The desired worm $w = w(v) = (X', X'', Y', Y'')$ on Φ is assigned by the following three rules (see Fig. 5 where the corresponding worms are drawn in bold and $\delta \neq 0$). We denote the ℓ_1-distance of points A, A' by $\|AA'\|$.

(8.8) The horizontal limb of w connects the points $X' := (2a_1, a_2)$ and $X'' := (2a_3, a_2)$ (degenerating into the single point $X = (2a_1, a_2)$ when $\delta \leq 0$; cf. (8.1)).

(8.9) Let $\delta \geq 0$. Then (see Fig. 5(i),(ii),(iii)):

(i) if $\ell < a_2$, then w is the VH-worm in which the lower limb connects the points $Y' := (2a_1, \ell)$ and $Y'' = X' = (2a_1, a_2)$;

(ii) if $a_2 \leq \ell \leq a_2 + 2\delta$, then w is the H-worm in which the point Y is located at $(2a_1 + \ell - a_2, a_2)$;

(iii) if $\ell > a_2 + 2\delta$, then w is the HV-worm in which the upper limb connects the points $Y' = X'' = (2a_3, a_2)$ and $Y'' := (2a_3, \ell - 2\delta)$.

(8.10) Let $\delta \leq 0$. Then w is the V-worm and (see Fig. 5(iv),(v)):

(i) if $\ell \leq a_2 + \delta$, then $Y' := (2a_1, \ell)$ and $Y'' := (2a_1, a_2 + |\delta|) = X + (0, |\delta|)$;

(ii) if $\ell > a_2 + \delta$, then $Y' := (2a_1, a_2 + \delta) = X + (0, \delta)$ and $Y'' := (2a_1, \ell + 2|\delta|)$.

First of all we have to check that each worm constructed by (8.8)–(8.10) is well-defined and their set is complete (coincides with the set of worms in $W(\bar{c})$).

By (8.8), both points X', X'' lie in Φ and their first coordinates are even (as required in (7.2)(i)). Also any horizontal line-segment in Φ connecting points (x', y) and (x'', y) with x', x'' even and y integer is present as the horizontal limb of $w(v)$ for some $v \in S$ (namely, with $a_1(v) = x'/2$, $a_2(v) = y$, $a_3(v) = x''/2$). To verify other properties, it is useful to partition S into subsets (*groups*) $S(a, \delta)$, each depending on a pair (a, δ) as in (8.1) and consisting of all $v \in S$ such that $a(v) = a$ and $\delta(v) = \delta$.

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That is, $S(a, \delta)$ is formed by the vertices of the corresponding middle subcrystal, denoted as $P(a, \delta)$; so $P(a, \delta) = P(v)$ for all $v \in S(a, \delta)$. We consider two cases.

Case I: $\delta \geq 0$. Define $Q = Q(a, \delta)$ to be the union of the vertical line-segments $Y_0 X'$ and $X'' Y_1$ and the horizontal line-segment $X'X''$, where $Y_0 := (2a_1, 0)$ and $Y_1 := (2a_3, c_2)$. Note that the sum of lengths of these segments is $a_2 + (2a_3 - 2a_1) + (c_2 - a_2) = c_2 + 2\delta$ (in view of $a_3 - a_1 = \delta$); this is equal to the length $c_2^{\uparrow\downarrow}$ of the path $P = P(a, \delta)$ (cf. (8.7)). Regarding Q as the corresponding path from Y_0 to Y_1, we can identify it with P. Comparing (8.4) with (8.9), we observe that: the first vertex of P (where $\ell = 0$) is identified with Y_0, and the last vertex of P (where $\ell = c_2 + 2\delta$) with Y_1. For the first vertex, the arising worm w has the Y' point at Y_0 and is the largest VH-worm for $S(a, \delta)$, whereas for the last vertex, w has the Y'' point at Y_1 and is the largest HV-worm for $S(a, \delta)$. When moving along P step by step, the current worm $w = w(v)$ evolves as follows: while $\ell(v) < a_2$, w is a VH-worm whose lower limb $Y'X'$ shortens by 1 at each step; while $a_2 \leq \ell(v) < a_2 + 2\delta$, w is an H-worm in which the point Y shifts to the right by 1 at each step; and while $a_2 + 2\delta \leq \ell(v) < c_2 + 2\delta$, w is an HV-worm whose upper limb $X''Y''$ increases by 1 at each step. This behavior matches the action of operator $\widetilde{\mathbf{2}}$ on the worm graph.

Case II: $\delta \leq 0$. Let $Y_0 := (2a_1, 0)$ and $Y_1 := (2a_1, c_2)$ (cf. the previous case) and define $\widetilde{Y}_0 := (2a_1, a_2 + \delta)$ and $\widetilde{Y}_1 := (2a_1, a_2 + |\delta|)$. (Recall that $\delta \leq 0$ implies $a_1 = a_3$.) Comparing (8.4) with (8.10), we observe that: for the first vertex of $P = P(a, \delta)$ (where $\ell = 0$), the arising worm w has the Y' point at Y_0 and the Y'' point at $\widetilde{Y}_1$, whereas for the last vertex of P (where $\ell = c_2 + 2\delta$), w has Y' at $\widetilde{Y}_0$ and Y'' at Y_1. When moving along P, the current H-worm evolves as follows: while $\ell(v) < a_2 + \delta$ (and therefore, $\|Y'X\| > |\delta| = \|XY''\|$), the lower limb $Y'X$ shortens by 1 at each step and Y'' rests at $\widetilde{Y}_1$, and while $a_2 + \delta \leq \ell(v) < c_2 + 2\delta$ (and therefore, $\|Y'X\| = |\delta| \leq \|XY''\|$), the upper limb XY'' increases by 1 at each step and Y' rests at $\widetilde{Y}_0$. (Note that $\min\{\|Y'X\|, \|XY''\|\}$ is invariant and equal to $|\delta|$.) Again, this matches the action of $\widetilde{\mathbf{2}}$ on the worm graph.

Thus, we come to the following

Proposition 8.3 *By the above construction, the correspondence $v \mapsto w(v)$ is a bijection between the vertices of the symmetric extract B from $K(c)$ and the vertices of the worm graph $W = W(\bar{c}_1, \bar{c}_2)$. Under this bijection, the edges of second color $\bar{2}$ of B are transferred to the edges of second color $\widetilde{2}$ of W.*

8.3 Verification of edges of color 1

To finish the proof of Theorem 6.3 it remains to show that under the above correspondence $v \mapsto w(v)$, the edges of color $\bar{1}$ in the symmetric extract B from $K = K(c_1, c_2, c_1)$ are transferred one-to-one to the edges of color $\widetilde{1}$ in the worm graph $W(c_1, c_2)$. This involves additional ideas and technical tools.

Proposition 8.4 *For each $v \in S$, the following properties hold:*

(i) *if $\widetilde{\mathbf{1}}$ does not act at the worm $w(v)$, then $\bar{\mathbf{1}}$ does not act at v;*

(ii) *if $\widetilde{\mathbf{1}}$ acts at $w(v)$, then $\bar{\mathbf{1}}$ acts at v and $w(\bar{\mathbf{1}}v) = \widetilde{\mathbf{1}}w(v)$.*

Proof We will use induction on the *length* of $w(v)$, which is defined below. When needed, handling one or another object related to v (or another vertex in S), we will include v as argument in corresponding notation (on the other hand, we often omit v when it is clear from the context). We associate to v the integers $a_1(v)$, $a_2(v)$, $a_3(v)$, $\delta(v)$, $\ell(v)$ as before. Considering the worm $w(v)$ in the form of six-tuple $(x'(v), y(v), x''(v); y'(v), x(v), y''(v))$ as defined in Section 7, we introduce the following values:

$$p_1(v) := x'(v)/2, \qquad p_2(v) := y'(v), \qquad q_1(v) := x''(v)/2, \qquad (8.11)$$
$$q_2(v) := y''(v) \quad \text{and} \quad \eta(v) := q_1(v) + q_2(v) - p_1(v) - p_2(v).$$

Then $p_i \leq q_i$, $i = 1, 2$, and the corresponding points $X'(v)$, $X''(v)$, $Y'(v)$, $Y''(v)$ are located within the rectangle $R_v := \{(\alpha, \beta) \in \mathbb{R}^2 \colon 2p_1(v) \leq \alpha \leq 2q_1(v), p_2 \leq \beta \leq q_2(v)\}$. Moreover, at least one of $X'(v)$, $Y'(v)$ lies at the south-west corner $(2p_1(v), p_2(v))$ of R_v and at least one of $X''(v)$, $Y''(v)$ lies at the north-east corner $(2q_1(v), q_2(v))$; one may say that $w(v)$ *spans* R_v. We also call R_v the *domain* of $w(v)$. It degenerates into a horizontal segment (a vertical segment, a single point) when $w(v)$ is an H-worm (resp. a V-worm, a principal point).

In addition, extending $p(v)$ and $q(v)$ to the self-complementary triples $\widehat{p} = \widehat{p}(v) := (p_1(v), p_2(v), p_1(v))$ and $\widehat{q} = \widehat{q}(v) := (q_1(v), q_2(v), q_1(v))$, we consider the self-complementary vertices $\check{v}[\widehat{p}]$ and $\check{v}[\widehat{q}]$ in the principal lattice Π of K and define the graph B_v to be the interval of B from $\check{v}[\widehat{p}]$ to $\check{v}[\widehat{q}]$. We will use the following easy corollary from Proposition 2.3.

Corollary 8.5 *B_v is isomorphic to $B(q(v) - p(v))$ (the symmetric extract from $K(\widehat{q}(v) - \widehat{p}(v))$.* ∎

The number $\eta(v)$ in (8.11) is just what we call the *length* of $w(v)$. We assume by induction that the required properties (i),(ii) are valid for each $v' \in S$ with $\eta(v') < \eta(v)$. When $\widetilde{\mathbf{1}}$ acts at $w(v)$, we denote by u the element of S such that $w(u) = \widetilde{\mathbf{1}}w(v)$ (existing by Proposition 8.3). (So our goal in this case is to show that $u = \bar{\mathbf{1}}v$.) From the description of the worm model one can see that for the domain R_u, only three situations are possible: (a) $R_u \subset R_v$ (where the inclusion is strict), (b) $R_u \supset R_v$, and (c) $R_u = R_v$. The first case is easy.

Claim 1 (i) *If $R_u \subset R_v$, then $\bar{\mathbf{1}}$ acts at v and $u = \bar{\mathbf{1}}v$.* (ii) *Suppose $\bar{\mathbf{1}}$ acts at v and let $v' := \bar{\mathbf{1}}v$. If $\eta(v') < \eta(v)$, then $\widetilde{\mathbf{1}}$ acts at $w(v)$ and $u = v'$.*

Proof (i) Clearly $R_u \subset R_v$ implies $\eta(u) < \eta(v)$. Applying the induction to the vertex u in the reversed graph B^{rev} (which is isomorphic to B), one can conclude that the operator reverse to $\bar{\mathbf{1}}$ transfers u to v. Since $\bar{\mathbf{1}}$ is invertible, $u = \bar{\mathbf{1}}v$.

Part (ii) is proved in a similar way. ∎

The situation $R_u \supseteq R_v$ is less trivial. We examine three cases.

Case 1: $w(v)$ is an H-worm, i.e. $p_2 = q_2 =: r$ (hereinafter p_i stands for $p_i(v)$, and q_i for $q_i(v)$). Then $q_1 - p_1 = \delta \geq 0$, $X'(v) = (2p_1, r)$, $X''(v) = (2q_1, r)$ and $Y(v) = (x, r)$ for some $2p_1 \leq x \leq 2q_1$. For $i = 0, 1, \ldots, 2\delta$, let v_i denote the vertex in S such that $w(v_i)$ is the H-worm with $X'(v_i) = X'(v)$, $X''(v_i) = X''(v)$ and $Y(v_i) = (2p_1 + i, r)$. These worms have the same domain, and $v = v_j$ for some j.

If $j > \delta$, then $\|X'(v)Y(v)\| = j > 2\delta - j = \|Y(v)X''(v)\|$; therefore (cf. (7.4)) $\widetilde{\mathbf{1}}$ acts at $w(v)$ and moves the X' point by two units to the right. Then $R_u \subset R_v$, and we are done by Claim 1(i).

Now let $j \leq \delta$. We rely on the following claim; it will be proved in the Appendix.

Claim 2 *When $j \leq \delta$, operator $\bar{\mathbf{1}}$ acts at v if and only if $q_1 < c_1$.*

If $q_1 = c_1$ (and therefore $X''(v)$ lies on the right boundary of the entire region $R(c_1, c_2)$), then $\widetilde{\mathbf{1}}$ does not act at $w(v)$. By Claim 2, $\bar{\mathbf{1}}$ does not act at v as well, and we obtain (i) in the proposition.

Thus, we may assume that $q_1 < c_1$. This and $\|X'(v)Y(v)\| = j \leq 2\delta - j = \|Y(v)X''(v)\|$ imply that $\widetilde{\mathbf{1}}$ acts at $w(v)$ and moves the X'' point by two units to the right. Then $w(u) = \widetilde{\mathbf{1}}w(v)$ is the H-worm with $X'(u) = X'(v)$, $Y(u) = Y(v)$ and $X''(u) = X''(v) + (2, 0) = (2q_1 + 2, r)$. In particular, $R_u \supset R_v$.

By Claim 2, $\bar{\mathbf{1}}$ acts at v; let $v' := \bar{\mathbf{1}}v$. Notice that the assertion in this claim depends on $\delta - j$ and $c_1 - q_1$, but not on p_1, q_1, r, c_1. So we can apply it to the subgraph B_u of B (the interval between the principal vertices $\check{v}[(p_1, r, p_1)]$ and $\check{v}[(q_1 + 1, r, q_1 + 1)]$), appealing to Corollary 8.5. This implies the important fact that the *vertex v' belongs to B_u*, whence $w(v')$ is an H-worm with $R_{v'} \subseteq R_u$. Let

$$X'(v') = (2p_1', r), \quad X''(v) = (2q_1', r) \quad \text{and} \quad Y(v') = (2p_1 + j', r).$$

We have to show that $v' = u$, i.e.

$$p_1' = p_1, \quad q_1' = q_1 + 1 \quad \text{and} \quad j' = j. \tag{8.12}$$

To show this, we will construct certain routes in B_u and appeal to the fact that B_u is graded (i.e. for any route P in B_u and each color ζ, the difference $k_\zeta(P)$ between the number of forward and backward edges of color ζ in P depends only on the beginning and end of P; cf. 1 in Section 6). (Hereinafter by a *route* we mean a path with possible backward edges.)

The case $\eta(v') < \eta(v)$ is impossible, by Claim 1(ii) and the inclusion $R_u \supset R_v$. Hence either (a) $p_1' = p_1$ and $q_1' = q_1$, or (b) $p_1' = p_1 + 1$ and $q_1' = q_1 + 1$, or (c $p_1' = p_1$ and $q_1' = q_1 + 1$. First of all we exclude (a) and (b).

1) Suppose $p_1' = p_1$ and $q_1' = q_1$. Then the worm $w(v')$ is obtained from $w(v)$ by moving the Y point by $j' - j$ units (to the right or left depending on the sign of $j' - j$). Hence $w(v') = \widetilde{\mathbf{2}}^{j'-j}w(v)$, implying $v' = \bar{\mathbf{2}}^{j'-j}v$ (since the "operator" w and the second color commute, by Proposition 8.3). The latter is impossible since $v' = \bar{\mathbf{1}}v$ and B_u is graded.

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2) Suppose $p_1' = p_1 + 1$ and $q_1' = q_1 + 1$. We can find in B_u a route P_1 from $\check{v}_0 := \check{v}[(p_1, r, p_1)]$ to v and a route P_2 from v to $\check{v}_1 := \check{v}[(q_1 + 1, r, q_1 + 1)]$ such that

$$k_{\tilde{\mathbf{1}}}(P_1) = \delta, \quad k_{\tilde{\mathbf{2}}}(P_1) = j, \quad k_{\tilde{\mathbf{1}}}(P_2) = \delta + 2, \quad k_{\tilde{\mathbf{2}}}(P_2) = 2\delta + 2 - j. \tag{8.13}$$

To get P_1, we construct a sequence of worms as follows. Starting with the principal worm $w(\check{v}_0)$, we first move the X'' point (by two units per move) δ times, obtaining $X'' = (2p_1 + 2\delta, r)$. Then move Y (by one unit per move) j times, obtaining $Y = (2p_1 + j, r)$. And to get P_2, we start with $w(v)$ and first make $2\delta - j$ moves with Y (obtaining $Y = X'' = (2q_1, r)$), then q_1 moves with X' (obtaining $X' = (2q_1, r)$), and finally one move with X'', two moves with Y and one move with X'. This results in $X' = X'' = Y = (2q_1 + 2, r)$, giving the worm $w(\check{v}_1)$. Each of these moves on worms induces passing through the corresponding edge in B_u: this is clear when operator $\tilde{\mathbf{2}}$ applies and follow by the induction when $\tilde{\mathbf{1}}$ applies (since at least one of the two worms involved in the operation has the length less than $\eta(v)$). This gives P_1, P_2 satisfying (8.13). (Note that the construction of P_2 remains correct when $\eta(v) = 0$, i.e. $p_1 = q_1$ and $v = \check{v}_0$. In this case there is a unique directed path P from $\check{v}_0$ to $\check{v}_1 = \check{v}[(p_1 + 1, r, p_1 + 1)]$ in B; see the illustrations of $K(1, 0, 1)$ and $B(1, 0)$ in Section 6. It corresponds to the relation $w(\check{v}_1) = \widetilde{\mathbf{1}}\widetilde{\mathbf{2}}\widetilde{\mathbf{2}}\widetilde{\mathbf{1}} w(\check{v}_0)$ in W.)

Using a similar procedure, we construct a route Q_2 from v' to $\check{v}_1$ such that

$$k_{\tilde{\mathbf{1}}}(Q_2) = \delta \quad \text{and} \quad k_{\tilde{\mathbf{2}}}(Q_2) = 2\delta + 2 - j'.$$

Then concatenating the route P_1, the move by $\bar{\mathbf{1}}$ from v to v', and the route Q_2, we obtain a route P' from $\check{v}_0$ to $\check{v}_1$ for which $k_{\tilde{\mathbf{1}}}(P') = \delta + 1 + \delta$. On the other hand, the path from $\check{v}_0$ to $\check{v}_1$ that is the concatenation of P_1 and P_2, denoted as $P_1 \cdot P_2$, gives $k_{\tilde{\mathbf{1}}}(P_1 \cdot P_2) = 2\delta + 2$. A contradiction with the gradedness of B.

3) Now let $p_1' = p_1$ and $q_1' = q_1 + 1$. To show the desired equality $j' = j$, consider the vertex z in B_u such that $w(z)$ is the H-worm with

$$X'(z) = (2p_1 + 2, r), \quad X''(z) = (2q_1 + 2, r) \quad \text{and} \quad Y(z) = (2q_1 + 2 - j, r).$$

Then the isomorphism between B_u^{rev} and B_u swaps z and v (as well as $\check{v}_0$ and $\check{v}_1$). This implies that $\bar{\mathbf{1}}^{-1}$ acts at z and the worm of $z' := \bar{\mathbf{1}}^{-1} z$ should be "symmetric" to the worm of v', namely:

$$X'(z') = (2p_1, r), \quad X''(z') = (2q_1 + 2, r) \quad \text{and} \quad Y(z') = (2q_1 + 2 - j', r).$$

Then $w(v')$ is transformed into $w(z')$ by moving the Y point by $(2\delta + 2 - j') - j' = 2\delta + 2 - 2j'$ units. Now concatenating the above route P_1 from $\check{v}_0$ to v, the $\bar{\mathbf{1}}$-edge from v to v', the corresponding route from v' to z', the $\bar{\mathbf{1}}$-edge from z' to z, and the route from z to $\check{v}_1$ that is "symmetric" to P_1, we obtain a route Q from $\check{v}_0$ to $\check{v}_1$ such that

$$k_{\tilde{\mathbf{2}}}(Q) = j + (2\delta + 2 - 2j') + j = 2\delta + 2 + 2j - 2j'.$$

Since B is graded, we must have $k_{\bar{\mathbf{2}}}(Q) = k_{\bar{\mathbf{2}}}(P_1) + k_{\bar{\mathbf{2}}}(P_2) = 2\delta + 2$ (cf. (8.13)). Hence $j' = j$, yielding (8.12). Thus, we obtain (ii) in the proposition.

Case 2: $w(v)$ is a V-worm with $X(v) \neq Y''(v)$. Then R_v is the vertical segment connecting the points $Y'(v) = (f, p_2)$ and $Y''(v) = (f, q_2)$, where $f := p_1 = q_1$. Also $X(v) = (f, p_2 + j)$ for some $0 \leq j < h := q_2 - p_2$. For $i = 0, \ldots, h$, let z_i denote the vertex of B such that $R_{z_i} = R_v$ and $X(z_i) = (f, p_2 + i)$; then $v = z_j$.

By (7.4), for $i = 0, \ldots, h - 1$, the worm $w(z_i)$ is transformed by $\widetilde{\mathbf{1}}$ into the worm $w(z_{i+1})$. Define $z_i' := \bar{\mathbf{1}} z_i$. Our goal is to show that $z_i' = z_{i+1}$ for $i = 0, \ldots, h - 1$. We rely on the following claim which will be proved in the Appendix.

Claim 3 *Operator $\bar{\mathbf{1}}$ acts at each z_i with $0 \leq i < h$.*

This claim does not impose any conditions on f, p_2, c_1, c_2; so it is applicable to the subgraph B_v of B (taking into account Corollary 8.5). This implies $R_{z_i'} \subseteq R_{z_i} = R_v$ for each $i < h$. The strict inclusion here is excluded by Claim 1(ii). Hence $R_{z_i'} = R_v$, implying $z_i' = z_{i'}$ for some $i' \in \{0, \ldots, h\}$.

To show the desired equality $i' = i + 1$ for each i, we use the next claim, denoting the principal vertices $\check{v}[(f, p_2, f)]$ and $\check{v}[(f, q_2, f)]$ by $\check{v}_0$ and $\check{v}_1$, respectively.

Claim 4 *Let $i \neq h/2$. Let P_1 be a route (in B) from $\check{v}_0$ to z_i, and P_2 a route from z_i to $\check{v}_1$. Then $k_{\bar{\mathbf{1}}}(P_1) = i$ and $k_{\bar{\mathbf{1}}}(P_2) = h - i$. In particular, $k_{\bar{\mathbf{1}}}(P_1 \cdot P_2) = h$.*

Proof We construct a sequence of worms starting with $w(z_i)$ and ending with $w(\check{v}_1)$ (aiming to obtain P_2 as above). Two cases are possible: (a) $i > h/2$, and (b) $i < h/2$. In case (a), we first apply to $w(z_i)$ operator $\widetilde{\mathbf{2}}$ which, in view of $\|Y'(z_i)X(z_i)\| = i > h - i = \|X(z_i)Y''(z_i)\|$, moves Y' by one unit up (obtaining $Y' = (f, p_2 + 1)$). Then we make $h - i$ moves with X (obtaining $X = (f, q_2)$), followed by $h - 1$ moves with Y'. This results in $Y' = Y'' = X = (f, q_2)$. And in case (b), we first apply operator $\widetilde{\mathbf{2}}^{-1}$ which, in view of $\|Y'(z_i)X(z_i)\| < \|X(z_i)Y''(z_i)\|$, moves Y'' by one unit down (obtaining $Y'' = (f, q_2 - 1)$). Then we make $h - i - 1$ moves with X (obtaining $X = (f, q_2 - 1)$), followed by $h - 1$ moves with Y' (obtaining $Y' = (f, q_2 - 1)$), and finally make one move with each of Y'', X, Y' (in this order); this results in $Y' = Y'' = X = (f, q_2)$.

Since every time we either apply operator $\widetilde{\mathbf{2}}$ or $\widetilde{\mathbf{2}}^{-1}$, or apply $\widetilde{\mathbf{1}}$ to a worm $w(z')$ with $R_{z'} \subset R_v$, we can conclude that in each case the constructed sequence of worms induces a route P from z_i to $\check{v}_1$ in B_v. The equality $k_{\bar{\mathbf{1}}}(P) = h - i$ follows by counting the number of applications of $\widetilde{\mathbf{1}}$ in the sequence.

A route P' from $\check{v}_0$ to z_i, giving $k_{\bar{\mathbf{1}}}(P') = i$, is constructed in a similar way (by applying the above procedure to z_i and $\check{v}_0$ in B_v^{rev}). ∎

Consider $i \neq h/2, h$. If $i' \neq h/2$, take a route P_1 from $\check{v}_0$ to z_i and a route P_2' from $z_{i'}$ to $\check{v}_1$. Let Q be the concatenation of P_1, the $\bar{\mathbf{1}}$-edge from z_i to $z_{i'}$, and P_2'. By Claim 4, $k_{\bar{\mathbf{1}}}(P_1) = i$ and $k_{\bar{\mathbf{1}}}(P_2') = h - i'$; therefore, $k_{\bar{\mathbf{1}}}(Q) = i + 1 + h - i'$. On the other hand, $k_{\bar{\mathbf{1}}}(Q)$ must be equal to h (since B is graded and $k_{\bar{\mathbf{1}}}(P_1 \cdot P_2) = h$). This implies $i' = i + 1$, as required.

Now let $i = h/2 - 1$ (when h is even). Then the case $i' = h/2$ ($= i + 1$) is only possible (since $i' \neq h/2$ would imply $i' \neq i + 1$, leading to a contradiction with the argument above). Finally, let $i = h/2$. Considering B_v^{rev} and $i'' = h/2 + 1$, we have $\bar{1}^{-1} z_{i''} = z_{h/2}$. Hence $i' = i'' = h/2 = i + 1$, as required.

Thus, (ii) in the proposition is valid for each $v = z_i$ with $i < h$.

Case 3: $w(v)$ is a proper VH-worm or a proper HV-worm or a V-worm with $X(v) = Y''(v) \neq Y'(v)$. (This is the simplest case in our analysis.) When $w(v)$ is an HV-worm, we have $p_1 < q_1$, $X'(v) = (2p_1, p_2)$ and $X''(v) = Y'(v) = (2q_1, p_2)$. Then operator $\widetilde{1}$ moves X' to $(2p_1 + 2, p_2)$; this gives $R_u \subset R_v$, and (ii) in the proposition follows from Claim 1(i).

So we may assume that $w(v)$ is the VH-worm with $Y'(v) = (2p_1, p_2)$, $Y''(v) = X'(v) = (2p_1, q_2)$ and $X''(v) = (2q_1, q_2)$; it degenerates into a V-worm when $p_1 = q_1$. The following claim will be proved in the Appendix.

Claim 5 _When $w(v)$ is a VH-worm, $\bar{1}$ acts at v if and only if $q_1 < c_1$._

In case $q_1 = c_1$, we have (i) in the proposition.

Let $q_1 < c_1$. Then $\widetilde{1}$ acts at $w(v)$ and moves X'' by two units to the right, making the VH-worm $w(u)$ with $Y'(u) = Y'(v)$ and $X''(u) = (2q_1 + 2, q_2)$. Claim 5 is applicable to B_u, whence the vertex $v' := \bar{1}v$ satisfies $R_{v'} \subseteq R_u$. The case $R_{v'} \neq R_u$ is excluded (by repeating some reasonings from Case 1 and using Claim 1(ii)).

Let $R_{v'} = R_u$. Then $w(v')$ is either the VH-worm $w(u)$ (yielding (ii) in the proposition), or the HV-worm with $X'(v') = (2p_1, p_2)$ and $Y''(v') = (2q_1 + 2, q_2)$. One easily shows that the latter is impossible.

Thus, the proposition is valid in all cases. ∎

This completes the proofs of Theorems 6.3 and 6.1.

Remark 5. Return to a symmetric A_{2n-1}-crystal $K = K(c)$ and its symmetric extract $B = B(\bar{c})$. Let $S\mathcal{B}(c)$ be the set of tuples $a \in \mathcal{B}(c)$ satisfying $a_{2n-i} = a_i$. Then $S\Pi := \{\check{v}[a] : a \in S\mathcal{B}(c)\}$ is the set of self-complementary principal vertices in K. The projection $a \mapsto (a_1, \ldots, a_n) =: \bar{a}$ gives a bijection between $S\Pi$ and the integer n-box $\mathcal{B}(\bar{c})$. Also for any $a, a' \in S\mathcal{B}(c)$ with $a \leq a'$, the interval $I(a, a')$ of K between the principal vertices $\check{v}[a]$ and $\check{v}[a']$ is isomorphic to the symmetric A_{2n-1}-crystal $K(a' - a)$; this implies that the symmetric extract from $I(a, a')$ is isomorphic to the B_n-crystal $B(\bar{a}' - \bar{a})$. Thus, $S\Pi$ possesses properties similar to (P1)–(P2) mentioned in the Introduction for A-crystals, due to which this set can be regarded as the _principal lattice_ of the B_n-crystal B. Note, however, that $S\Pi$ need not satisfy property (P3). This is seen already for $n = 2$. Indeed, in this case any principal vertex $\check{v}$ is represented by a single point (x, y) (a principal worm) in the worm model; therefore, in the subcrystal of color 1 (color 2) containing $\check{v}$, all vertices correspond to horizontal (resp. vertical) worms covering (x, y), and none of proper VH- or HV-worms can be used.

A similar construction of principal lattices can be given for C-crystals.

9 Deriving C_n-crystals from symmetric A_{2n}-crystals

By an analogue with the construction and results in Sections 6–8, we can construct and examine the n-colored graphs being symmetric extracts from symmetric A_{2n}-crystals. In this section we show that these graphs are C_n-crystals.

Using terminology and notation similar to those in Section 6, we consider an A_{2n}-crystal $K = K(c) = (V(K), E_1 \sqcup \ldots \sqcup E_{2n})$ with a parameter c satisfying $c_i = c_{2n+1-i}$, denote by $\bar{i}$ the pair of complementary colors $(i, i' := 2n + 1 - i)$ for $i = 1, \ldots, n$, and consider the corresponding complementarity involution $\sigma : V(K) \to V(K)$ and the set S of self-complementary vertices $v = \sigma(v)$ of K. When $i < n$, the colors i and i' are not neighboring, and, as before, we draw an edge of color $\bar{i}$ from a vertex $u \in S$ to a vertex $v \in S$ if and only if $v = F_i F_{i'}(u)$ $(= F_{i'} F_i(u))$. The edges of color $\bar{n} = (n, n' = n + 1)$ are assigned in a different way: we draw an $\bar{n}$-edge from $u \in S$ to $v \in S$ if and only if $v = F_n F_{n'} F_{n'} F_n(u)$ (see an explanation in Remark 6 below).

Let $C = (S, E_{\bar{1}} \sqcup \ldots \sqcup E_{\bar{n}})$ be the obtained n-colored graph. One can see that a majority of reasonings of Section 6 remain applicable to our case, yielding common properties of C and B.

Proposition 9.1 *The symmetric extract C from a symmetric A_{2n}-crystal $K(c)$ satisfies axioms (BC1) and (BC2) (from Section 2.2).*

Thus, like the A_{2n-1} case, the key problem is to characterize the components of the two-colored subgraph of C with colors $\overline{n-1}$ and $\bar{n}$. This is equivalent to characterizing the extract $(S, E_{\bar{1}} \sqcup E_{\bar{2}})$ from a symmetric A_4-crystal $K(c)$ with colors 1,2,3,4. Our goal is to show the following

Theorem 9.2 *Let $K(c)$ be a symmetric A_4-crystal. Then the symmetric extract C from K is the C_2-crystal with parameter $\bar{c} = (c_1, c_2)$ respecting the Cartan coefficients $m_{12} = -1$ and $m_{21} = -2$.*

(Cf. Theorem 6.3.) The proof will consist of several stages, which are analogous, to some extent, to those in Section 8. We will identify the vertices of C with certain quadruples (somewhat different from those in the $A_3 \to B_2$ reduction), describe the polytope spanned by these quadruples and show their one-to-one correspondence to the vertices of the worm graph $W(c_2, c_1)$. This gives a counterpart of Proposition 8.3 (but with colors $\widetilde{1}, \widetilde{2}$ swapped). Theorem 9.2 will be obtained by comparing the edges in both graphs. As a result, the desired relation between A- and C-crystals follows.

Corollary 9.3 (Cf. [12, Th. 3.2.4].) *The symmetric extract from a symmetric A_{2n}-crystal is a C_n-crystal, and any C_n-crystal is obtained in this way.*

Figure 6 illustrates the "simplest" symmetric A_4-crystals $K(1, 0, 0, 1)$ and its symmetric extract $C(1, 0)$.
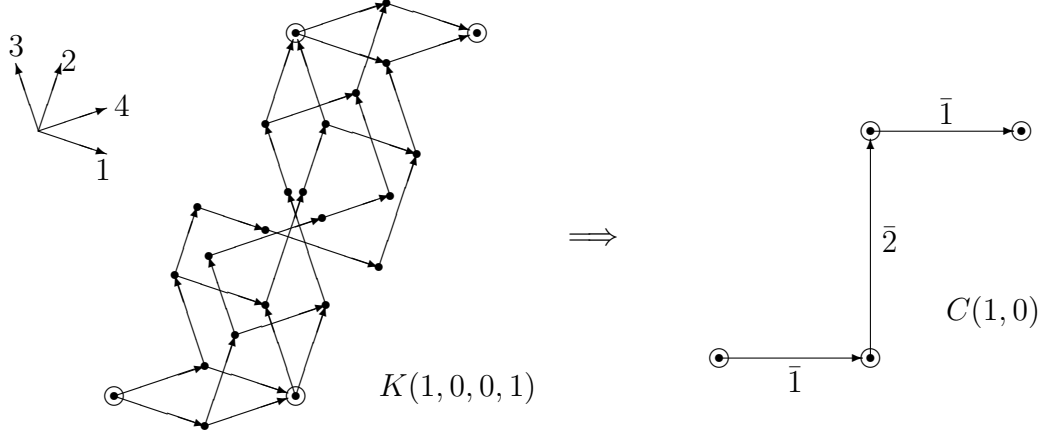
Figure 6: Creation of $C(1,0)$.

9.1 A polyhedral description for the vertices of C in case $n = 2$

Consider a self-complementary vertex $v \in S$. Take the corresponding objects related to v: the upper subcrystal $K^{\uparrow}[a]$ (with colors $1, 2, 3$) and the lower subcrystal $K^{\downarrow}[b]$ (with colors $2, 3, 4$), the corresponding deviations $\Delta = (\Delta_1, \Delta_2, \Delta_3)$ and $\nabla = (\nabla_2, \nabla_3, \nabla_4)$, the hearts $z := v^{\uparrow}[a|\Delta]$ and $z' := v^{\downarrow}[b|\nabla]$, and the middle subcrystal $K^{\uparrow\downarrow}$ (with colors $2, 3$). (So $K^{\uparrow\downarrow}$ contains v and has the heart z in $\Pi^{\uparrow}[a]$, and z' in $\Pi^{\downarrow}[b]$.) Denoting $\delta := \Delta_1$, we observe that

$$\Delta = (\delta, 0, -\delta) \qquad \text{and} \qquad \nabla = (-\delta, 0, \delta). \tag{9.1}$$

(Cf. Lemma 8.1(iii).) Indeed, by the complementarity, we have $(\nabla_2, \nabla_3, \nabla_4) = (\Delta_3, \Delta_2, \Delta_1)$. This together with the relations $\nabla_i = -\Delta_{i-1}$ for $i = 2, 3, 4$ (by (3.2)) give $\Delta_1 = \nabla_4 = -\Delta_3 = -\nabla_2$ and $\Delta_2 = \nabla_3 = -\Delta_2$, yielding (9.1).

We also have $b = (b_1, b_2, b_3, b_4) = (a_4, a_3, a_2, a_1)$ (by the complementarity). Comparing this with the relations $b_1 = a_1 + \Delta_1^+$ and $b_2 = a_2 + \Delta_2^+ + \Delta_1^-$ (by (3.1)) and using (9.1), we obtain

$$a_4 = a_1 + \delta^+ \qquad \text{and} \qquad a_3 = a_2 + \delta^-. \tag{9.2}$$

Thus, a and Δ (as well as b, ∇) are determined by a_1, a_2, δ. The latter triple obeys:

$$0 \le a_1 \le c_1; \qquad 0 \le a_2 \le c_2; \tag{9.3}$$
$$-a_2 \le \delta \le c_1 - a_1. \tag{9.4}$$

Here (9.3) is identical to (8.2) (and is obvious), but (9.4) is somewhat different from (8.3). To see (9.4), take the parameter $c^{\uparrow}$ and the heart coordinate $\hbar^{\uparrow}$ of $K^{\uparrow}[a]$; they are $c_i^{\uparrow} = c_i - a_i + a_{i+1}$ and $\hbar_i^{\uparrow} = a_{i+1}$ for $i = 1, 2, 3$ (cf. (2.8),(2.9)). Since $\Delta = (\delta, 0, -\delta)$, the evident relations $-\hbar_i^{\uparrow} \le \Delta_i \le c_i^{\uparrow} - \hbar_i^{\uparrow}$, $i = 1, 2, 3$, are reduced to

$$-a_2 \le \delta \le c_1 - a_1 \qquad \text{and} \qquad -a_4 \le -\delta \le c_3 - a_3.$$

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The left expression is exactly (9.4) (and both inequalities in it are essential), whereas the right expression is redundant (since (9.2) implies $a_4 \geq a_1 + \delta \geq \delta$ and $a_3 - \delta \leq a_2 \leq c_2 = c_3$).

(An additional fact is that (9.2),(9.3),(9.4) imply $0 \leq a \leq c$. This is because $0 \leq a_1 \leq a_4 = a_1 + \delta^+ \leq c_1 = c_4$ and $0 \leq a_2 + \delta^- = a_3 \leq a_2 \leq c_2 = c_3$.)

One more, fourth, ingredient in the description of v comes up when we consider the location of v in the middle subcrystal $K^{\uparrow\downarrow}$ (this is a symmetric A_2-crystal, and our analysis becomes more involved compared with the $A_3 \rightarrow B_2$ case where the corresponding middle subcrystal is a path). We rely on the following (using terminology from Section 5.1).

Lemma 9.4 $K^{\uparrow\downarrow}$ *is symmetric, and a vertex v of $K^{\uparrow\downarrow}$ belongs to S if and only if v lies on the diagonal $D = \{(i,i)\}$ of the critical lattice $\Pi^{\uparrow\downarrow}$ of $K^{\uparrow\downarrow}$. In particular, $K^{\uparrow\downarrow}$ contains exactly $|D|$ self-complementary vertices.*

Proof Take a path in $K^{\uparrow\downarrow}$ going from v to the sink $\widetilde{t}$ of $K^{\uparrow\downarrow}$, and let P' be the path in K going from v and complementary to P. Then P' is contained in $K^{\uparrow\downarrow}$ (since the edges of both P, P' have colors only 2 and 3). Since the end $\widetilde{t}$ of P has no outgoing edges of colors 2 and 3, so does the end $\sigma(\widetilde{t})$ of P'. This implies $\sigma(\widetilde{t}) = \widetilde{t}$, i.e. $\widetilde{t} \in S$. Similarly, considering a path in $K^{\uparrow\downarrow}$ going from the source $\widetilde{s}$ of $K^{\uparrow\downarrow}$ to v (or to $\widetilde{t}$), we can conclude that $\widetilde{s} \in S$.

Consider a vertex u of $K^{\uparrow\downarrow}$. If u belongs to the critical lattice $\Pi^{\uparrow\downarrow}$ and has coordinates (α, β) in it, then u is expressed as $(\mathbf{23})^\beta (\mathbf{32})^\alpha (\widetilde{s})$. Then the complementary vertex $\sigma(u)$ is $(\mathbf{32})^\beta (\mathbf{23})^\alpha (\widetilde{s})$, and $u = \sigma(u)$ holds if and only if $\alpha = \beta$, i.e. u lies on the diagonal D. Also when $u = \widetilde{t}$, the pair $(\alpha, \beta = \alpha)$ becomes the parameter of $K^{\uparrow\downarrow}$; so $K^{\uparrow\downarrow}$ is symmetric. Finally, if u is not in $\Pi^{\uparrow\downarrow}$ and belongs to a right (left) sail, then $\sigma(u)$ must belong to the complementary left (resp. right) sail, whence $\sigma(u) \neq u$. ∎

Remark 6. For two consecutive elements $u = (i,i)$ and $w = (i+1, i+1)$ of D, we have $w = \mathbf{2332}(u)$. Also $\mathbf{32}(u) \neq \mathbf{23}(u)$. This leads to the following consequence of Lemma 9.4: if u, w are two self-complementary vertices of a symmetric A_{2n}-crystal K and if w is obtained from u by applying a string of operators F_n and F_{n+1}, then $w = (F_n F_{n+1}^2 F_n)^q (u)$ for some integer $q \geq 0$. This justifies the definition of edges of color $\bar{n}$ in the n-colored extract C.

Remark 7. For a vertex u of the A_4-crystal K and a color i, define $\varepsilon_i(u) := h_i(u) - t_i(u)$, where $h_i(u)$ and $t_i(u)$ are the lengths of the maximal paths in K having color i and going from u and to u, respectively (cf. Section 2.1). For a vertex u of C and a color $\bar{i}$, the values $h_{\bar{i}}, t_{\bar{i}}$ and $\varepsilon_{\bar{i}}(u)$ are defined in a similar way. Considering an edge (u, w) of color $\bar{2}$ in C, we have (using axiom (A2) for K):

$$\varepsilon_{\bar{1}}(u) = \varepsilon_1(u) = \varepsilon_1(\mathbf{2}(u)) - 1 = \varepsilon_1(\mathbf{32}(u)) - 1$$
$$= \varepsilon_1(\mathbf{332}(u)) - 1 = \varepsilon_1(\mathbf{2332}(u)) - 2 = \varepsilon_{\bar{1}}(w) - 2,$$

in view of $w = \mathbf{2332}(u)$ and $u, w \in S$. In its turn, an edge (u, w) of color $\bar{1}$ in C gives

$$\varepsilon_{\bar{2}}(u) = \varepsilon_2(u) = \varepsilon_2(\mathbf{1}(u)) - 1 = \varepsilon_2(\mathbf{41}(u)) - 1 = \varepsilon_{\bar{2}}(w) - 1.$$

As a consequence for n-colored extracts, we obtain the relation $\varepsilon_{n\bar{-}1}(u)-\varepsilon_{n\bar{-}1}(w)=-2$ for an edge (u,w) of color $\bar{n}$, and the relation $\varepsilon_{\bar{n}}(u)-\varepsilon_{\bar{n}}(w)=-1$ for an edge (u,w) of color $n\bar{-}1$. This hints that the pair of colors $n\bar{-}1$, $\bar{n}$ behaves as prescribed by the Cartan coefficients $m_{n-1,n}$ and $m_{n,n-1}$ for C_n-crystals figured in Axiom (BC4′).

Return to v as above. Using (3.4), we express the parameter $c^{\uparrow\downarrow}$ of $K^{\uparrow\downarrow}$ by

$$c_2^{\uparrow\downarrow} = c_2 - \Delta_2 + \Delta_1 = c_2 + \delta \qquad \text{and} \qquad c_3^{\uparrow\downarrow} = c_2^{\uparrow\downarrow}. \tag{9.5}$$

By Lemma 9.4, the vertex v is a point (ρ,ρ) in the diagonal D. Then (9.5) implies that the number ρ satisfies

$$0 \le \rho \le c_2 + \delta. \tag{9.6}$$

The obtained quadruple $(a_1(v), a_2(v), \delta(v), \rho(v))$ determining a vertex $v \in S$ is just what we call the *description* of v in the graph C with the "first" color $\bar{1}=(1,4)$ and the "second" color $\bar{2}=(2,3)$. It satisfies (9.3),(9.4),(9.6). It is straightforward to check the corresponding converse property, and we can conclude with the following polyhedral result (cf. Theorem 8.2 for the A_3 case).

Theorem 9.5 *The above correspondence $v \mapsto (a_1, a_2, \delta, \rho)$ gives a bijection between the set of self-complementary vertices of a symmetric A_4-crystal $K(c)$ and the set of integer solutions to the linear system $\{(9.3),(9.4),(9.6)\}$.*

9.2 Relation to the worm graph

Based on the above linear system, we can associate the vertices of C to worms, but now the colors $\widetilde{1}$ and $\widetilde{2}$ in the worm model turn out to be "swapped": they become related to colors $\bar{2}$ and $\bar{1}$ in C, respectively (which is agreeable with Remark 7).

More precisely, we should deal with the reversed parameter $c' = (c_1', c_2') := (c_2, c_1)$ and accordingly consider the worm graph $W(c')$ whose vertices (worms) live in the rectangle $R(c') = \{(\alpha, \beta): 0 \le \alpha \le 2c_2,\ 0 \le \beta \le c_1\}$. Given $v \in S$ and its corresponding a, δ, ρ, we construct the worm $w = w(v) = (X', X'', Y', Y'')$ as follows:

$$Y' := (a_2 + a_3,\ a_1); \qquad Y'' := (a_2 + a_3,\ a_4); \tag{9.7}$$
$$X' := (2\min\{\rho, a_3\},\ a_1 + \min\{(\rho - a_3)^+, \delta^+\}); \tag{9.8}$$
$$X'' := (2a_2 + 2(\rho - a_2 - \delta)^+,\ a_1 + \min\{(\rho - a_3)^+, \delta^+\}). \tag{9.9}$$

These settings look cumbersome, and to make them more comprehensible, let us define the following points on $R(c')$:

$$I_0 := (0, a_1), \quad J_0 := (2a_3, a_1), \quad J_1 := (2a_2, a_4), \quad I_1 := (2c_2, a_4).$$

Then, using relations (9.2)–(9.4),(9.6), we can rewrite (9.7)–(9.9) in a more enlightening form, given in (9.10) and (9.11).

(9.10)   Let $\delta \le 0$. Then $a_1 = a_4$, $a_3 = a_2 + \delta$, and w is an H-worm such that:

(i) Y is the middle point of the horizontal segment J_0J_1;

(ii) if $\rho \leq a_3$, then X' is the point $(2\rho, a_1)$ occurring in the horizontal segment I_0J_0, and $X'' = J_1$;

(iii) if $\rho \geq a_3$, then $X' = J_0$, and X'' is the point $(2a_2 + 2(\rho - a_3), a_1)$ occurring in the horizontal segment J_1I_1.

(See Fig. 7(a),(b).) Here (i) is obvious. Part (ii) follows from $\min\{\rho, a_3\} = \rho$, $\min\{(\rho - a_3)^+, \delta^+\} = 0$ and $\rho - a_2 - \delta = \rho - a_3 \leq 0$ (since $a_3 = a_2 + \delta^-$); cf. (9.8). And (iii) follows from $\min\{\rho, a_3\} = a_3$, $\min\{(\rho - a_3)^+, \delta^+\} = 0$ and $(\rho - a_2 - \delta)^+ = \rho - a_3$; cf. (9.9).

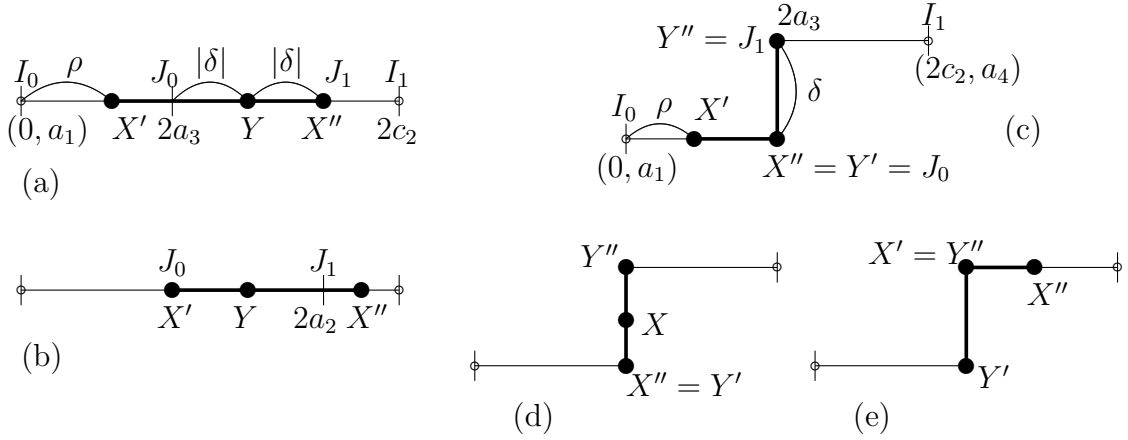Figure 7: (a) $\delta < 0$, $\rho < a_3$; (b) $\delta < 0$, $\rho > a_3$; (c) $\delta > 0$, $\rho < a_3$; (d) $\delta > 0$, $a_3 < \rho < a_3 + \delta$; (e) $\delta > 0$, $\rho > a_3 + \delta$.

(9.11)   Let $\delta \geq 0$. Then $a_4 = a_1 + \delta$, $a_3 = a_2$, and w is an HV- or V- or VH-worm such that:

(i) J_0J_1 is the vertical limb of w, $Y' = J_0 = (2a_3, a_1)$ and $Y'' = J_1 = (2a_3, a_4)$;

(ii) if $\rho \leq a_3$, then w is an HV-worm, X' is the point $(2\rho, a_1)$ occurring in the horizontal segment I_0J_0, and $X'' = J_0$;

(iii) if $a_3 \leq \rho \leq a_3 + \delta$, then w is a V-worm, and X is the point $(2a_3, a_1 + \rho - a_3)$ occurring in the vertical limb J_0J_1;

(iv) if $\rho \geq a_3 + \delta$, then w is a VH-worm, $X' = J_1$, and X'' is the point $(2a_3 + 2(\rho - a_3 - \delta), a_4)$ in the horizontal segment J_1I_1.

(See Fig. 7(c),(d),(e).) Again, (i) is obvious. Part (ii) is provided by $\min\{\rho, a_3\} = \rho$, $\min\{(\rho - a_3)^+, \delta^+\} = 0$ and $\rho - a_2 - \delta \leq 0$. Part (iii) follows from $\min\{\rho, a_3\} = a_3$, $\min\{(\rho - a_3)^+, \delta^+\} = \rho - a_3$ and $\rho - a_2 - \delta \leq 0$. And (iv) follows from $\min\{\rho, a_3\} = a_3$, $a_1 + \min\{(\rho - a_3)^+, \delta^+\} = a_1 + \delta = a_4$ and $(\rho - a_2 - \delta)^+ = \rho - a_3 - \delta$.

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Relations (9.2)–(9.4),(9.6) ensure that $w(v)$ is indeed a feasible worm on $R(c')$, and one can see that any worm in $W(c')$ can be obtained by the above construction from some $v \in S$. Furthermore, for an edge (u,v) of color $\bar{2}$ in C, we have $a(v) = a(u)$, $\delta(v) = \delta(u)$ and $\rho(v) = \rho(u) + 1$. Considering (9.10),(9.11), one can realize that the worm $w(v)$ is obtained from $w(u)$ by applying the *first* operator on $W(c')$ (which updates X', X'' and preserves Y', Y'').

Thus, we can conclude with the following result analogous to Proposition 8.3.

Proposition 9.6 *By the above construction, the correspondence $v \mapsto w(v)$ is a bijection between the vertices of the symmetric extract C from an A_4-crystal $K(c_1, c_2, c_2, c_1)$ and the vertices of the worm graph $W = W(c_2, c_1)$. Under this bijection, the edges of color $\bar{2}$ in C are transferred one-to-one to the edges of color $\widetilde{1}$ in W.*

9.3 Verification of edges of color 2

It remains to prove that the edges of color $\bar{1}$ in C are bijective to the edges of color $\widetilde{2}$ in $W = W(c_2, c_1)$.

Proposition 9.7 *For each $v \in S$, the following properties hold:*

(i) *if $\widetilde{2}$ does not act at the worm $w(v)$, then $\bar{1}$ does not act at v;*

(ii) *if $\widetilde{2}$ acts at $w(v)$, then $\bar{1}$ acts at v and $w(\bar{1}v) = \widetilde{2}w(v)$.*

Proof Our approach is similar to used in Section 8.3; this allows us to argue in a more concise manner, omitting details which can be restored by the similarity.

Given $v \in S$, we define

$$p_1(v) := a_3(v), \quad q_1(v) := a_2(v), \quad p_2(v) := a_1(v), \quad q_2(v) := a_4(v),$$

consider $\eta(v)$ and R_v as before and proceed by induction on the length $\eta(v)$ of the worm $w(v)$, using Corollary 8.5 (where, instead of B_v, one should consider the interval C_v of C between the principal vertices $\check{v}[p_2(v), p_1(v), p_1(v), p_2(v)]$ and $\check{v}[q_2(v), q_1(v), q_1(v), q_2(v)]$). When $\widetilde{2}$ acts at $w(v)$, we define u so that $w(u) = \widetilde{2}w(v)$. As before, either $R_u \subset R_v$ or $R_u \supseteq R_v$ takes place.

The role of Claim 1 is now played by the following claim whose proof is similar.

Claim 1′ (i) *If $R_u \subset R_v$, then $\bar{1}$ acts at v and $u = \bar{1}v$.* (ii) *If $\bar{1}$ acts at v and if $\eta(v') < \eta(v)$, where $v' := \bar{1}v$, then $\widetilde{2}$ acts at $w(v)$ and $u = v'$.* ∎

Next we examine three cases ("symmetric" to the ones in the proof of Proposition 8.4); here p_i, q_i, a_i concern v.

Case 1: $w(v)$ is a V-worm, i.e. $p_1 = q_1 =: r$. Then $q_2 - p_2 = \delta \geq 0$. Also $Y'(v) = (2r, p_2)$, $Y''(v) = (2r, q_2)$ and $X(v) = (2r, y)$ for some $p_2 \leq y \leq q_2$. Let $j := y - p_2$ (then $0 \leq j \leq \delta$).

If $j > \delta/2$, then $\|Y'(v)X(v)\| = j > \delta - j = \|X(v)Y''(v)\|$. Hence (cf. (7.5)) operator $\widetilde{\mathbf{2}}$ acts at $w(v)$ and moves Y' by one unit up. This gives $R_u \subset R_v$, and we are done by Claim 1$'$(i).

Now let $j \le \delta/2$. The following claim will be proved in the Appendix.

Claim 2$'$ *When $j \le \delta/2$, operator $\bar{\mathbf{1}}$ acts at v if and only if $q_2 < c_1$.*

In case $q_2 = c_1$, neither operator $\bar{\mathbf{1}}$ acts at v (by Claim 2$'$), nor operator $\widetilde{\mathbf{2}}$ acts at $w(v)$ (as Y'' cannot be moved upward). So we obtain (ii) in the proposition.

Thus, we may assume that $q_2 < c_1$. This and $\|Y'(v)X(v)\| = j \le \delta - j = \|X(v)Y''(v)\|$ imply that $\widetilde{\mathbf{2}}$ acts at $w(v)$ and moves Y'' by one unit up. Then $w(u)$ is the V-worm with $Y'(u) = (2r, p_2)$, $Y'' = (2r, q_2 + 1)$ and $X(u) = (2r, p_2 + j)$.

Since Claim 2$'$ is applicable to the subgraph C_u of C, the vertex $v' := \bar{\mathbf{1}}v$ belongs to C_u, and $w(v')$ is a V-worm with $R_{v'} \subseteq R_u$. Let

$$Y'(v') = (2r, p_2'), \quad Y''(v') = (2r, q_2') \quad \text{and} \quad X(v') = (2r, p_2 + j').$$

To show that $p_2' = p_2$, $q_2' = q_2 + 1$ and $j' = j$ (yielding $v' = u$), we argue as follows. Since $\eta(v') < \eta(v)$ is impossible (by Claim 1$'$(ii) and in view of $\eta(u) > \eta(v)$), we are in one of the following three cases: (a) $p_2' = p_2$ and $q_2' = q_2$; (b) $p_2' = p_2 + 1$ and $q_2' = q_2 + 1$; (c) $p_2' = p_2$ and $q_2' = q_2 + 1$.

Case (a) is impossible. For otherwise we would have $w(v') = \widetilde{\mathbf{1}}^{j'-j}w(v)$, implying $v' = \bar{\mathbf{2}}^{j'-j}v$ (since $\bar{\mathbf{2}}$ on C corresponds to $\widetilde{\mathbf{1}}$ on W, by Proposition 9.6.) But C is graded and $v' = \bar{\mathbf{1}}v$.

In cases (b) and (c), acting as in the proof of Proposition 8.4, we can construct four routes P_1, P_2, Q_1, Q_2, respectively, from $\check{v}_0 := \check{v}[p_2, r, r, p_2]$ to v, from v to $\check{v}_1 := \check{v}[q_2 + 1, r, r, q_2 + 1]$, from $\check{v}_0$ to v', and from v' to $\check{v}_1$. Moreover, these routes are consistent with W, in the sense that each of their $\bar{\mathbf{1}}$-edges ($\bar{\mathbf{2}}$-edges) induces a $\widetilde{\mathbf{2}}$-edge (resp. $\widetilde{\mathbf{1}}$-edge) in W. A direct count (using the corresponding routes in W) gives

$$k_{\bar{\mathbf{1}}}(P_1) = \delta, \quad k_{\bar{\mathbf{2}}}(P_1) = j, \quad k_{\bar{\mathbf{1}}}(P_2) = \delta + 2, \quad k_{\bar{\mathbf{2}}}(P_2) = \delta + 1 - j. \tag{9.12}$$

In case (b), a similar count for Q_2 gives $k_{\bar{\mathbf{1}}}(Q_2) = q_2' - p_2' = \delta$. Then concatenating the route P_1, the $\bar{\mathbf{1}}$-edge from v to v', and the route Q_2, we obtain a route P' from $\check{v}_0$ to $\check{v}_1$ such that $k_{\bar{\mathbf{1}}}(P') = 2\delta + 1$. But $k_{\bar{\mathbf{1}}}(P_1 \cdot P_2) = 2\delta + 2$; a contradiction.

Thus, case (c) is only possible. To show that $j = j'$, take the vertices z, z' in C_u whose worms are "symmetric" to $w(v), w(v')$, respectively, i.e.

$$Y'(z) = (2r, p_2 + 1), \quad Y''(z) = (2r, q_2 + 1), \quad X(z) = (2r, q_2 + 1 - j);$$
$$Y'(z') = (2r, p_2), \quad Y''(z') = (2r, q_2 + 1), \quad X(z') = (2r, q_2 + 1 - j').$$

By the isomorphism between C_u^{rev} and C_u, we have $z' = \bar{\mathbf{1}}^{-1}z$. Also $w(v')$ is transformed into $w(z')$ by moving the X point by $(q_2 + 1 - j') - (p_2 + j') = \delta + 1 - 2j'$ points (regarding the upward direction as positive). Now concatenating the route P_1 from $\check{v}_0$ to v, the $\bar{\mathbf{1}}$-edge from v to v', the corresponding route from v' to z', the $\bar{\mathbf{1}}$-edge

from z' to z, and the route from z to $\check{v}_1$ "symmetric" to P_1, we obtain a route Q from $\check{v}_0$ to $\check{v}_1$ such that

$$k_{\bar{\mathbf{2}}}(Q) = j + (\delta + 1 - 2j') + j = \delta + 1 + 2j - 2j'.$$

On the other hand, $k_{\bar{\mathbf{2}}}(P_1 \cdot P_2) = \delta + 1$ (see (9.12)). Hence $j' = j$, as required.

<u>*Case 2*</u>: $w(v)$ is an H-worm with $Y(v) \neq X''(v)$. Then R_v is the horizontal segment connecting the points $X'(v) = (2p_1, f)$ and $X''(v) = (2q_1, f)$, where $f := p_2 = q_2$. Also $Y(v) = (2p_1 + j,\, f)$ for some $0 \leq j < 2h$, where $h := |\delta| = q_1 - p_1$. For $i = 0, \ldots, 2h$, let z_i denote the vertex of C such that $R_{z_i} = R_v$ and $Y(z_i) = (2p_1 + i,\, f)$; then $v = z_j$.

By (7.5), for $0 \leq i < 2h$, operator $\widetilde{\mathbf{2}}$ transforms the worm $w(z_i)$ into $w(z_{i+1})$. Define $z_i' := \bar{\mathbf{1}} z_i$. Our goal is to show that $z_i' = z_{i+1}$ for $i = 0, \ldots, 2h - 1$. The following claim will be proved in the Appendix.

Claim 3′ *Operator $\bar{\mathbf{1}}$ acts at each z_i with $0 \leq i < 2h$.*

This claim is applicable to the subgraph C_v of C, implying that $R_{z_i'} \subseteq R_{z_i} = R_v$ for each $i < 2h$. The strict inclusion here is impossible by Claim 1′(ii); so $R_{z_i'} = R_v$ and $z_i' = z_{i'}$ for some $i' \in \{0, \ldots, 2h\}$.

The following claim is proved analogously to the proof of Claim 4 in Section 8.3. Here $\check{v}_0 := \check{v}[f, p_1, p_1, f]$ and $\check{v}_1 := \check{v}[f, q_1, q_1, f]$.

Claim 4′ *Let $i \neq h$. Let P_1 be a route (in C) from $\check{v}_0$ to z_i, and P_2 a route from z_i to $\check{v}_1$. Then $k_{\bar{\mathbf{1}}}(P_1) = i$ and $k_{\bar{\mathbf{1}}}(P_2) = 2h - i$. In particular, $k_{\bar{\mathbf{1}}}(P_1 \cdot P_2) = 2h$.* ▐

This claim enables us to prove the desired equalities $i' = i + 1$. If $i \neq h, 2h$ and $i' \neq h$, then concatenating a route from $\check{v}_0$ to z_i, the $\bar{\mathbf{1}}$-edge from z_i to $z_{i'}$, and a route $z_{i'}$ to $\check{v}_1$, we obtain a route Q from $\check{v}_0$ to $\check{v}_1$ with $k_{\bar{\mathbf{1}}}(Q) = i + 1 + 2h - i'$ (by Claim 4′ applied to i and i'). This gives $i' = i + 1$ (since $k_{\bar{\mathbf{1}}}(Q)$ must be equal to $2h$).

If $i = h - 1$, then $i' = h\ (= i + 1)$ is only possible. And if $i = h$, then $i' = h + 1$ follows by considering C_v^{rev} and $i'' := h + 1$ (obtaining $\bar{\mathbf{1}}^{-1} z_{i''} = z_h$, whence $i' = i''$).

<u>*Case 3*</u>: $w(v)$ is a proper VH-worm or a proper HV-worm or an H-worm with $Y(v) = X''(v) \neq X'(v)$. This case is examined in a similar way as Case 3 from the proof of Proposition 8.4, relying on the following claim proved in the Appendix.

Claim 5′ *When $w(v)$ is an HV-worm, $\bar{\mathbf{1}}$ acts at v if and only if $q_2 < c_1$.*

This completes the proofs of Proposition 9.7, Theorem 9.2 and Corollary 9.3. ▐▐▐

Appendix. Proofs of claims

In this section we prove the claims from Propositions 8.4 and 9.7 that were left without verification, thus completing the proofs of main theorems on B- and C-crystals from Sections 6,8,9. In the proofs below we will extensively use the explicit formulas on the parameters, heart coordinates and etc. in A-crystals and their subcrystals.

A.1 Proof of Claims 2, 3, 5 from Proposition 8.4

Recall that in Section 8 we considered a self-complementary vertex v of a symmetric A_3-crystal $K = K(c_1, c_2, c_1)$ and its symmetric extract B. Let $a = (a_1, a_2, a_3)$, $\Delta = (\Delta_1, \Delta_2) = (\delta, -\delta)$, ℓ, $c^\uparrow = (c_1^\uparrow, c_2^\uparrow)$, $\hbar^\uparrow = (\hbar_1^\uparrow, \hbar_2^\uparrow)$, $c^{\uparrow\downarrow} = (c_2^{\uparrow\downarrow})$, $\hbar^{\uparrow\downarrow} = (\hbar_2^{\uparrow\downarrow})$ be the corresponding objects concerning v and its related upper subcrystal $K^\uparrow[a]$ and middle subcrystal $K^{\uparrow\downarrow}$ (path of color 2). They are subject to relations (8.1)–(8.7).

The path $K^{\uparrow\downarrow}$, further denoted as P_2, is the *lower* subcrystal containing v in the A_2-crystal $K^\uparrow[a]$ with colors 1,2. The heart z of P_2 (i.e. the common vertex of P_2 and the critical lattice $\Pi^\uparrow$ of $K^\uparrow[a]$) has the coordinate $\hbar_2^{\uparrow\downarrow}$ in P_2 (counted from the beginning of P_2) equal to $a_2 + \delta$, by (8.7). Hence the deviation of v from z in P_2 is expressed as

$$\Delta_2^{\uparrow\downarrow} := \ell - \hbar_2^{\uparrow\downarrow} = \ell - a_2 - \delta. \tag{10.1}$$

The maximal path P_1 of color 1 that contains v is an *upper* subcrystal in $K^\uparrow[a]$. The location of v in P_1 is crucial for the claims that we are going to prove: operator $\bar{\mathbf{1}}$ acts at v if and only if v is *not the last vertex of P_1* (taking into account that either none or both of operators $1, 3$ act at v). In order to find this location, we will compute the parameter (length) $c_1^{\uparrow\uparrow}$ of P_1, the coordinate $\hbar_1^{\uparrow\uparrow}$ of the heart z' of P_1 in P_1 itself, and the deviation $\Delta_1^{\uparrow\uparrow}$ of v from z' in P_1. Then

(10.2) $\bar{\mathbf{1}}$ acts at v if and only if $\hbar_1^{\uparrow\uparrow} + \Delta_1^{\uparrow\uparrow} < c_1^{\uparrow\uparrow}$.

The values figured in (10.2) can be expressed as follows (relying on the fact that P_1, P_2 and v are interrelated upper, lower and middle subcrystals in $K^\uparrow[a]$). The coordinates (locus) $b^\uparrow = (b_1^\uparrow, b_2^\uparrow)$ of z in $\Pi^\uparrow$ are expressed as $b^\uparrow = \hbar^\uparrow + \Delta$. Then (8.6) and the relations $\Delta = (\delta, -\delta)$ and $a_3 = a_1 + \delta^+$ give

$$b_1^\uparrow = a_2 + \delta \quad \text{and} \quad b_2^\uparrow = a_3 - \delta = a_1 + \delta^+ - \delta = a_1 - \delta^-. \tag{10.3}$$

Applying Theorem 3.1 to $K^\uparrow[a]$, P_1, P_2, we observe that the deviation $\Delta_1^{\uparrow\uparrow}$ is equal to $-\Delta_2^{\uparrow\downarrow}$, and the coordinates (locus) $a^\uparrow = (a_1^\uparrow, a_2^\uparrow)$ of z' in Π are expressed as $a_1^\uparrow = b_1^\uparrow + \Delta_2^{\uparrow\downarrow -}$ and $a_2^\uparrow = b_2^\uparrow + \Delta_2^{\uparrow\downarrow +}$ (cf. (3.3)). Using (10.1), we have

$$\Delta_1^{\uparrow\uparrow} = -\Delta_2^{\uparrow\downarrow} = a_2 + \delta - \ell; \tag{10.4}$$

$$a_1^\uparrow = b_1^\uparrow + \Delta_2^{\uparrow\downarrow -} = a_2 + \delta + (\ell - a_2 - \delta)^- = \min\{\ell, a_2 + \delta\}; \tag{10.5}$$

$$a_2^\uparrow = b_2^\uparrow + \Delta_2^{\uparrow\downarrow +} = a_1 - \delta^- + (\ell - a_2 - \delta)^+. \tag{10.6}$$

Also (cf. (2.8),(2.9)):

$$c_1^\uparrow = c_1 - a_1 + a_2, \quad c_1^{\uparrow\uparrow} = c_1^\uparrow - a_1^\uparrow + a_2^\uparrow \quad \text{and} \quad \hbar_1^{\uparrow\uparrow} = a_2^\uparrow. \tag{10.7}$$

Now (10.3)–(10.7) enable us to precisely compute the desired quantity:

$$c_1^{\uparrow\uparrow} - \hbar_1^{\uparrow\uparrow} - \Delta_1^{\uparrow\uparrow} = (c_1^\uparrow - a_1^\uparrow + a_2^\uparrow) - a_2^\uparrow - (a_2 + \delta - \ell)$$
$$= (c_1 - a_1 + a_2) - \min\{\ell, a_2 + \delta\} - a_2 - \delta + \ell$$
$$= c_1 - a_1 - \delta + \ell - \min\{\ell, a_2 + \delta\} =: \omega.$$

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Thus (by (10.2)), $\bar{\mathbf{1}}$ acts at v if and only if $\omega > 0$. (Note that simultaneously ω is equal to the length of the maximal $\bar{1}$-colored path from v in B, i.e. to $h_{\bar{1}}(v)$, using notation from Section 2.)

Now we are ready to prove Claims $2, 3, 5$.

1) The condition $j \leq \delta$ in the hypotheses of Claim 2 is equivalent to $\ell \leq a_2 + \delta$ (this is seen by considering the actions of operator $\widetilde{\mathbf{2}}$ on W described in Case I of Section 8.2). Hence (in view of $a_1 + \delta = a_3$)

$$\omega = c_1 - a_1 - \delta + \ell - \ell = c_1 - (a_1 + \delta) = c_1 - a_3 = c_1 - q_1,$$

proving Claim 2.

2) Similar conditions $\delta \geq 0$ and $\ell \leq a_2 + \delta$ hold in Claim 5, and we again obtain $\omega = c_1 - q_1$.

3) Since $w(v)$ in Claim 3 is a V-worm, we have $\delta \leq 0$ and $\ell \geq a_2 + \delta$ (see Case II in Section 8.2), whence $\min\{\ell, a_2 + \delta\} = a_2 + \delta$. Also $c_1 \geq a_1$. Then

$$\omega = c_1 - a_1 - \delta + \ell - a_2 - \delta \geq \ell - a_2 - 2\delta.$$

Observe that $X(v)$ is the point (a_1, a_2) and $Y''(v)$ is the point $(a_1, \ell - 2\delta)$. Now $\omega > 0$ follows from the condition that $X(v)$ lies below $Y''(v)$. ∎

A.2 Proof of Claims $2', 3', 5'$ from Proposition 9.7

In Section 9.3 we considered a self-complementary vertex v of an A_4-crystal $K = K(c_1, c_2, c_2, c_1)$ and related a, δ, ρ. Compared with the previous case, we are now forced to handle more subcrystals of K that contain v, namely, K', $K'^{\downarrow}$, $K'^{\uparrow}$, P_2, P_1, where:

(i) K' has colors 1,2,3 (it is just the upper subcrystal $K^{\uparrow}[a]$ of K);

(ii) $K'^{\downarrow}$ has colors 2,3; it is a lower subcrystal of K';

(iii) $K'^{\uparrow}$ has colors 1,2; it is an upper subcrystal of K';

(iv) P_2 has color 2; this path is an upper subcrystal of $K'^{\downarrow}$, a lower subcrystal of $K'^{\uparrow}$, and a middle subcrystal of K';

(v) P_1 has color 1; this path is an upper subcrystal of $K'^{\uparrow}$.

Accordingly we denote:

(vi) the principal lattices of $K, K', K'^{\downarrow}, K'^{\uparrow}$ by $\Pi, \Pi', \Pi'^{\downarrow}, \Pi'^{\uparrow}$, respectively;

(vii) the unique elements of $\Pi \cap K'$, $\Pi' \cap K'^{\downarrow}$, $\Pi' \cap K'^{\uparrow}$, $\Pi'^{\downarrow} \cap P_2$, $\Pi'^{\uparrow} \cap P_2$, $\Pi'^{\uparrow} \cap P_1$ by z, $z^{\downarrow}$, $z^{\uparrow}$, $z^{\downarrow\uparrow}$, $z^{\uparrow\downarrow}$, $z^{\uparrow\uparrow}$, respectively (these are the hearts of corresponding subcrystals);

(viii) the parameters of $K', K'^{\downarrow}, K'^{\uparrow}, P_1$ by $c', c'^{\downarrow}, c'^{\uparrow}, c'^{\uparrow\uparrow}$, respectively (each being a duly indexed vector; e.g., $c' = (c'_1, c'_2, c'_3)$, $c'^{\downarrow} = (c'^{\downarrow}_2, c'^{\downarrow}_3)$, $c'^{\uparrow\uparrow} = (c'^{\uparrow\uparrow}_2)$).

Considering one or another heart $z^{\bullet}$, we denote its coordinate in the principal lattice of the smaller subcrystal by $\hbar'^{\bullet}$; e.g., $\hbar'^{\downarrow}$ concerns $z^{\downarrow}$ in $\Pi'^{\downarrow}$, and $\hbar'^{\uparrow\downarrow}$ concerns $z^{\uparrow\downarrow}$ in P_2. Notation for additional objects (such as deviations, loci, et al.) will be specified on the way.

Like the previous case (cf. (10.2)), the following property is evident:

(10.8) $\bar{1}$ acts at v if and only if $c_1'^{\uparrow\uparrow} - \hbar_1'^{\uparrow\uparrow} - \Delta_1'^{\uparrow\uparrow} > 0,$

where $\Delta_1'^{\uparrow\uparrow}$ is the deviation of v from $z^{\uparrow\uparrow}$ in P_1. To express the quantity figured in (10.8) in terms of a_1, a_2, δ, ρ takes some technical efforts. We will use the following auxiliary values:

$$\phi := \rho - a_2 - \delta \qquad \text{and} \qquad \psi := \rho - a_2 - \delta^-. \tag{10.9}$$

Recall that the tuple $a = (a_1, a_2, a_3, a_4)$ (satisfying (9.2)) is the locus of the heart of K' in Π, and that the deviation in Π' of the heart $z^{\downarrow}$ of $K'^{\downarrow}$ from the heart $v[a]$ of K' is $\Delta = (\Delta_1, \Delta_2, \Delta_3) = (\delta, 0, -\delta)$ (by (9.1)).

Let $a', b', a'^{\uparrow}, b'^{\uparrow}$ denote the loci of $z^{\uparrow}$ in Π', of $z^{\downarrow}$ in Π', of $z^{\uparrow\uparrow}$ in $\Pi'^{\uparrow}$, of $z^{\uparrow\downarrow}$ in $\Pi'^{\uparrow}$, respectively. By (2.8) and (2.9) (applied to appropriate subcrystals), we have

$$c_1' = c_1 - a_1 + a_2, \qquad c_1'^{\uparrow} = c_1' - a_1' + a_2', \qquad c_1'^{\uparrow\uparrow} = c_1'^{\uparrow} - a_1'^{\uparrow} + a_2'^{\uparrow}; \tag{10.10}$$

$$\hbar_i' = a_{i+1} \;\; (i = 1, 2, 3), \qquad \hbar_i'^{\uparrow} = a_{i+1}' \;\; (i = 1, 2), \qquad \hbar_1'^{\uparrow\uparrow} = a_2'^{\uparrow}. \tag{10.11}$$

Since $b' = \hbar' + \Delta$, the first relation in (10.11) gives

$$b_1' = \hbar_1' + \delta = a_2 + \delta \quad \text{and} \quad b_2' = \hbar_2' + 0 = a_3 = a_2 + \delta^-. \tag{10.12}$$

By (10.12) and (2.11) (applied to $K', K'^{\downarrow}$), the locus $\hbar'^{\downarrow}$ of $z^{\downarrow}$ in $\Pi'^{\downarrow}$ is computed as

$$\hbar_2'^{\downarrow} = b_1' = a_2 + \delta \qquad \text{and} \qquad \hbar_3'^{\downarrow} = b_2' = a_2 + \delta^-. \tag{10.13}$$

Consider K' and its lower, upper and middle subcrystals containing v, namely, $K'^{\downarrow}, K'^{\uparrow}$ and P_2, respectively. Let $\Delta'^{\downarrow}$ denote the deviation of $z^{\downarrow\uparrow}$ from $z^{\downarrow}$ in $\Pi'^{\downarrow}$, and $\Delta'^{\uparrow}$ the deviation of $z^{\uparrow\downarrow}$ from $z^{\uparrow}$ in $\Pi'^{\uparrow}$. We know (cf. Lemma 9.4 and (9.5)) that the subcrystal $K'^{\downarrow}$ (with colors 2,3) is symmetric and has the parameter $c_i'^{\downarrow} = c_2 + \delta$ $(i = 2, 3)$. Also the vertex v is the point (ρ, ρ) in the principal lattice $\Pi'^{\downarrow}$. Hence v coincides with $z^{\downarrow\uparrow}$ (the heart of the path P_2 w.r.t. $\Pi'^{\downarrow}$). These facts give (using (10.9) and (10.13)):

$$\Delta_2'^{\downarrow} = \rho - \hbar_2'^{\downarrow} = \rho - a_2 - \delta = \phi \quad \text{and} \quad \Delta_3'^{\downarrow} = \rho - \hbar_3'^{\downarrow} = \rho - a_2 - \delta^- = \psi. \tag{10.14}$$

This enables us to compute $\Delta'^{\uparrow}$ and a'. Namely (using (3.2) and (3.3)):

$$\Delta_1'^{\uparrow} = -\Delta_2'^{\downarrow} = -\phi \qquad \text{and} \qquad \Delta_2'^{\uparrow} = -\Delta_3'^{\downarrow} = -\psi; \tag{10.15}$$

and

$$a_1' = b_1' + \Delta_2'^{\downarrow -} = a_2 + \delta + \phi^-; \tag{10.16}$$

$$a_2' = b_2' + \Delta_2'^{\downarrow +} + \Delta_3'^{\downarrow -} = a_2 + \delta^- + \phi^+ + \psi^-.$$

Also the locus $b'^{\uparrow}$ of $z^{\uparrow\downarrow}$ in $\Pi'^{\uparrow}$ is expressed (using (10.11) and (10.15)) as

$$b_1'^{\uparrow} = \hbar_1'^{\uparrow} + \Delta_1'^{\uparrow} = a_2' - \phi \quad \text{and} \quad b_2'^{\uparrow} = \hbar_2'^{\uparrow} + \Delta_2'^{\uparrow} = a_3' - \psi. \tag{10.17}$$

Next we use the fact that P_2, P_1 and v are the lower, upper and middle subcrystals of $K'^{\uparrow}$, respectively. The coordinate $\hbar_2'^{\uparrow\downarrow}$ of $z^{\uparrow\downarrow}$ in P_2 is equal to $b_1'^{\uparrow}$ (cf. (2.11)), and the coordinate of v in P_2 is equal to ρ (since the locus of $v = z^{\downarrow\uparrow}$ in $\Pi'^{\downarrow}$ is (ρ, ρ)). Hence the deviation $\Delta_2'^{\uparrow\downarrow}$ of v from $z^{\uparrow\downarrow}$ in P_2 is $\rho - b_1'^{\uparrow}$, and we have (using (10.15) and (10.17)):

$$\Delta_2'^{\uparrow\downarrow} = \rho - b_1'^{\uparrow} = \rho - a_2' + \phi = \rho - a_2 - \delta^- - \phi^+ - \psi^- + \phi = \phi^- + \psi^+. \qquad (10.18)$$

Finally, we have

$$\Delta_1'^{\uparrow\uparrow} = -\Delta_2'^{\uparrow\downarrow} \qquad \text{and} \qquad a_1'^{\uparrow} = b_1'^{\uparrow} + \Delta_2'^{\uparrow\downarrow^-} \qquad (10.19)$$

(cf. (3.2) and (3.3)), where $a'^{\uparrow}$ is the locus of $z^{\uparrow\uparrow}$ in $\Pi'^{\uparrow}$.

The obtained formulas enable us to compute the desired quantity:

$$
\begin{aligned}
c_1'^{\uparrow\uparrow} - \hbar_1'^{\uparrow\uparrow} - \Delta_1'^{\uparrow\uparrow} &= (c_1'^{\uparrow} - a_1'^{\uparrow} + a_2'^{\uparrow}) - a_2'^{\uparrow} - \Delta_1'^{\uparrow\uparrow} && \text{(by (10.10),(10.11))}\\
&= (c_1' - a_1' + a_2') - (b_1'^{\uparrow} + \Delta_2'^{\uparrow\downarrow^-}) + \Delta_2'^{\uparrow\downarrow} && \text{(by (10.10),(10.19))}\\
&= (c_1' - a_1' + a_2') - (a_2' - \phi) + \Delta_2'^{\uparrow\downarrow^+} && \text{(by (10.17))}\\
&= c_1' - a_1' + \phi + (\phi^- + \psi^+)^+ && \text{(by (10.18))}\\
&= (c_1 - a_1 + a_2) - (a_2 + \delta + \phi^-) + \phi + (\phi^- + \psi^+)^+ && \text{(by (10.10),(10.16))}\\
&= c_1 - a_1 - \delta + \phi^+ + (\phi^- + \psi^+)^+\\
&= c_1 - a_1 - \delta + (\rho - a_2 - \delta)^+ + ((\rho - a_2 - \delta)^- + (\rho - a_2 - \delta^-)^+)^+ =: \omega'.
\end{aligned}
$$

Now we are ready to prove Claims $2', 3', 5'$.

1) In the hypotheses of Claim $2'$, $w(v)$ is a V-worm; therefore, $\delta = a_4 - a_1 \geq 0$, $a_2 = a_3$ and $a_2 \leq \rho \leq a_2 + \delta$ (cf. (9.11)(iii)). Then $\phi = \rho - a_2 - \delta \leq 0$ and $\psi = \rho - a_2 \geq 0$. Also $j = \rho - a_2 \leq \delta/2$. We have

$$\omega' = c_1 - a_1 - \delta + (\rho - a_2 - \delta + \rho - a_2)^+ = c_1 - a_4 + (2\rho - 2a_2 - \delta)^+ = c_1 - a_4.$$

Since $a_4 = q_2$, $\omega' > 0$ if and only if $c_1 > q_2$, as required (in view of (10.8)).

2) In the hypotheses of Claim $3'$, $w(v)$ is an H-worm; therefore, $\delta \leq 0$. Moreover, $Y(v) \neq X''(v)$ implies $\delta < 0$ (cf. (9.10)). Also $a_1 \leq c_1$. Then $\omega' \geq c_1 - a_1 - \delta > 0$.

3) Since $w(v)$ in Claim $5'$ is an HV-worm, $\delta = a_4 - a_1 \geq 0$ and $\rho \leq a_3 = a_2$ (cf. (9.11)(ii)). Then $\phi \leq 0$ and $\psi \leq 0$. We have $\omega' = c_1 - a_1 - \delta + (\phi^- + \psi^+)^+ = c_1 - a_4 = c_1 - q_2$, as required. ∎

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